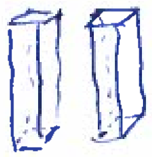
Simple Model System for Rod-like Particles

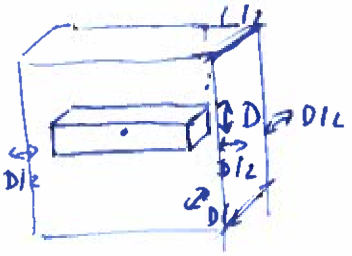
by J. de Graaf for PION 2023

(a) $B_{\alpha\alpha'} = B_{\alpha'\alpha}$ because the interaction potential is symmetric under exchange of particles

(b) (i)

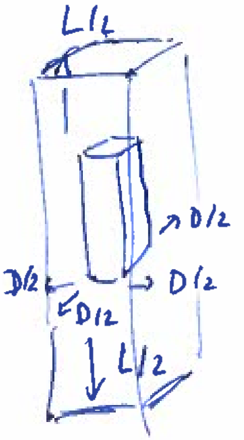
(ii)





block with side lengths

$(D+L) \times 2D \times (D+L)$



block with side lengths

$2D \times 2D \times 2L$

(c) class (ii) gives $B_{\parallel} = B_{11} = B_{22} = B_{33}$

$$B_{11} = -\frac{1}{2}\int_V d\bar{r}\,\underbrace{\left(\exp(-\beta\Phi_{11}(\bar{r}))-1\right)}$$

$= -1$ if overlap $\Phi_{11}(r) = \infty$

$= 0$ if no overlap

$\frac{1}{2}$(size of excluded volume)

(physically the excluded volume is carried for ½ by each partner in a pair)

$$B_{\parallel} = B_{11} = \frac{1}{2}(2D)(2D)(2L) = 4D^2L$$

$$B_{\perp} = B_{12} = \cdots = \frac{1}{2}(2D)(D+L)^2 = D(D+L)^2$$

d) $$\frac{B_{\parallel}}{L^2 D} = \frac{4D^2 L}{L^2 D} = \frac{4D}{L} \approx 0$$

$$\frac{B_{\perp}}{L^2 D} = \frac{D(L+D)^2}{L^2 D} = \left(1+\frac{D}{L}\right)^2 \approx 1$$

$$\psi = \frac{F L^2 D}{V k_B T} = L^2 D \left(\sum_{\alpha=1}^{3} \rho_\alpha \left(\log(\rho_\alpha V) - 1\right) + \sum_{\alpha=1}^{3} \sum_{\alpha'=1}^{3} B_{\alpha\alpha'} \rho_\alpha \rho_{\alpha'} \right)$$

$$= \left[\sum_{\alpha=1}^{3} c_\alpha \left(\log c_\alpha + \log \frac{V}{L^2 D} - 1 \right) + \sum_{\alpha=1}^{3} \sum_{\alpha'=1}^{3} \frac{B_{\alpha\alpha'}}{L^2 D} c_\alpha c_{\alpha'} \right]$$

$$\approx \sum_{\alpha=1}^{3} c_\alpha \left(\log c_\alpha - 1 + \log \frac{V}{L^2 D} \right) + 2 c_1 c_2 + 2 c_2 c_3 + 2 c_1 c_3 ,$$

because

$$\frac{B_{\alpha\alpha'}}{L^2 D} \approx \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

e)

disordered in position and in orientation

f)  we expect alignment, i.e., rotational symmetry is broken

g) nematic order parameter measures the amount in which the system is aligned with the broken symmetry axis, in this case $\hat{x}_3$

h) assume $c_3 = c$ then $1 = (1+2S)/3 \Rightarrow$

$3 - 1 = 2S \Rightarrow S = 1$ (fully nematic)

assume $c_3 = c_2 = c_1 \Rightarrow (1+2S)/3 = (1-S)/3$

$\Rightarrow 1 + 2S = 1 - S \Rightarrow S = 0$ (isotropic)

but you can also have $c_3 = 0 \quad c_1 = c_2$

$\Rightarrow S = -1/2$ (discotic)

So really $S \in [-1/2, 1]$.

i) Ignoring the thermal volume, we have

$$\left.\begin{aligned} \psi = \; & c_1 \log c_1 + c_2 \log c_2 + c_3 \log c_3 - c \\ & + 2c_1c_2 + 2c_2c_3 + 2c_1c_3 \\ c_1 = \; & \tfrac{c}{3}(1+2S), \quad c_2 = c_3 = \tfrac{c}{3}(1-S) \end{aligned}\right\} \Rightarrow$$

$$\frac{c}{3}(1+2S)\log\frac{c}{3}(1+2S) + \frac{2c}{3}(1-S)\log\left(\frac{c}{3}(1-S)\right) - c$$

$$+ \underset{2\times 2}{\frac{4c^2}{9}}\underbrace{(1+2S)(1-S)}_{(1+S-2S^2)} + \frac{2c^2}{9}\underbrace{(1-S)^2}_{1-2S+S^2}$$

$$= \frac{2c^2}{9}(3 - 3S^2) - c + (\text{log terms})$$

$$= c\left(\frac{2}{3}c(1-S^2)-1\right) + \frac{c}{3}\log\left(\frac{c}{3}(1+2S)\left(\frac{c}{3}(1-S)\right)^2\right)$$

$$+ \frac{2c}{3} S \log\left(\frac{1+2S}{1-S}\right) \Rightarrow \text{answer}$$

j) $$\frac{\partial\psi}{\partial S} = -\frac{4}{3}c^2 S + \frac{c}{3}\,\frac{2(1-S)^2-2(1-S)(1+2S)}{(1-S)^2(1+2S)}$$

$$+ \frac{2c}{3}\log\left(\frac{1+2S}{1-S}\right) - \frac{2}{3}cS\,\frac{1+2S}{1-S}\cdot\left(-\frac{1}{1+2S}-\frac{2(1-S)}{(1+2S)^2}\right)$$

$S=0 \Rightarrow$

$$\left.\frac{\partial\psi}{\partial S}\right|_{S=0} = 0 + \frac{c}{3}\,\frac{2-2\cdot1\cdot1}{1^2\cdot 1} + \frac{2c}{3}\log\left(\frac{1}{1}\right) - \frac{2}{3}c\cdot 0\cdot\ldots$$

$S=0$ gives the disordered phase, see (h)
or isotropic

k) $$\frac{\partial^2\psi}{\partial S^2} = -\frac{4}{3}c^2 + \frac{c}{3}\,\frac{\partial}{\partial S}\left[\frac{-6S}{(1-S)(1+2S)}\right]$$

$$+ \frac{2c}{3}\,\frac{1-S}{1+2S}\left(\frac{2}{1-S}+\frac{1+2S}{(1-S)^2}\right)$$

$$+ \frac{2c}{3}\,\frac{\partial}{\partial S}\left[S\,\frac{3}{(1-S)(1+2S)}\right]$$

are opposite!

$$= -\frac{4}{3}c^2 + 2c\,\frac{1}{(1+2S)(1-S)}$$

$$\left.\frac{\partial^2\psi}{\partial S^2}\right|_{S=0} = -\frac{4}{3}c^2 + 2c\,\frac{1}{1\cdot 1} = 2c - \frac{4}{3}c^2$$

from this we know we cross from a minimum to a maximum, whenever $\left.\frac{\partial^2\psi}{\partial S^2}\right|_{S=0} = 0 \Rightarrow 2c - \frac{4}{3}c^2 = 0$

$c=0 \;\vee\; c=3/2$ ← this one?

$$c^* = \frac{3}{2} \Rightarrow \quad \rho^* = \frac{3}{2DL^2}$$ for very long rods (high aspect ratio) this can be very small!

The phase we associate with $S \neq 0$ would be the nematic (orientationally ordered phase)

l) Phase coexistence requires:

- equal temperature $T^I = T^N$
- equal chemical potential $p^I = p^N$
- equal pressure $\mu^I = \mu^N$

- we implicitly have $T^I = T^N$
- for the pressure we realize that in virial theory

$$\beta p = \sum_\alpha \rho_\alpha + \sum_\alpha \sum_{\alpha'} B_{\alpha\alpha'} \rho_\alpha \rho_{\alpha'}$$

and due to the simple shape we have

$$\beta p = c + \frac{2}{3} c^2 (1 - S^2)$$ (relying on previous results)

makes sense as an ideal gas

- chemical potential we have via

$$\mu \propto \left(\frac{\partial \psi}{\partial c}\right) \Rightarrow \left(\frac{\partial \psi}{\partial c}\right)_{S^I, c^I} = \left(\frac{\partial \psi}{\partial c}\right)_{S^N, c^N}$$

we know $S^I = 0$ (because isotropic)

and we have $$\frac{\partial \psi}{\partial c} = \frac{4}{3} c (1 - S^2) \cancel{-1} - \frac{2}{3} S \log\left[\frac{1-S}{1+2S}\right] + \frac{1}{3} \log\left[\frac{c^3}{27}(1-S)^2(1+2S)\right] + \cancel{\frac{1}{3} c \cdot \frac{3}{c}}$$

So that

$$\frac{4}{3} c_I + \log \frac{c_I}{3} = \frac{4}{3} c_N (1 - S_N^2) - \frac{2}{3} S_N \log\left[\frac{1-S_N}{1+2S_N}\right] + \frac{1}{3} \log\left[\frac{c_N^3}{27}(1-S_N)^2(1+2S_N)\right]$$

from $p_I = p_N$ we get

$$c_I + \frac{2}{3}c_I^2 = c_N + \frac{2}{3}c_N^2(1-S_N^2)$$

unfortunately that leaves us with 2 equations in 3 unknowns, (c_I, c_N, S_N)

The resolution is that

$$\left(\frac{\partial\varphi}{\partial S}\right)_{c_N,S_N} = 0 \quad \& \quad \left(\frac{\partial^2\varphi}{\partial S^2}\right)_{c_N,S_N} > 0$$

we need to be in a minimum of the total system. From (j) we can infer

$$\frac{\partial\varphi}{\partial S} = -\frac{2}{3}c\left(2cS + \log\left[\frac{1-S}{1+2S}\right]\right) \quad \leftarrow \text{extra equation}$$

$$\frac{\partial^2\varphi}{\partial S^2} = -\frac{4}{3}c^2 + 2c\frac{1}{(1+2S)(1-S)} \quad (\text{use } k)$$

↖ simply a condition

m) ~~[illegible]~~

for $c^* > c > c_I$ metastable isotropic

for $c_N > c > c^*$ phase separated

for $c > c_N$ pure nematic

for $c < c_I$ pure isotropic

n) $c_N \approx 2$ $\quad \rho_N \approx \frac{2}{DL^2} = \left[\frac{2}{\frac{L}{100}L^2} = \frac{200}{L^3} \text{ (never mind)}\right]$

$V = D^2L$

$$\eta_N \approx \frac{2}{DL^2}\cdot D^2L = \frac{2D}{L} = \frac{1}{50} \quad \text{quite small}$$

Solution 1

$$
\begin{aligned}
{}^{14}_{7}N \;+\; {}^{14}_{7}N &\rightarrow {}^{24}_{12}Mg \;+\; {}^{4}_{2}He \\
{}^{14}_{7}N \;+\; {}^{14}_{7}N &\rightarrow {}^{16}_{8}O \;+\; {}^{12}_{6}C
\end{aligned}
$$

Solution 2

$$
\begin{aligned}
\Delta m &= m(products) - m(reactants) \\
E &= \Delta m c^2.
\end{aligned}
$$

This results in -17.2 MeV for reaction 1 and -10.5 MeV for reaction 2. The energy has a negative sign, meaning energy is released.

Solution 3

We find the following radii:

$$
\begin{aligned}
r({}^{14}N) &= 2.9 \times 10^{-15} \text{ m} \\
r({}^{24}Mg) &= 3.5 \times 10^{-15} \text{ m} \\
r({}^{4}He) &= 1.9 \times 10^{-15} \text{ m}.
\end{aligned}
$$

The Coulomb barrier for nuclei A and B is given by

$$
V_C = \frac{e^2}{4\pi\epsilon_0} \frac{Z_a Z_b}{r_a + r_b},
$$

resulting in $V_{C1} = 12$ MeV and $V_{C2} = 6.4$ MeV.

For the reaction to happen it is important that the product nuclei can emerge. Given that the total release of 17.2 MeV for reaction (1) is larger than the second barrier of 6.4 MeV (for the product nuclei), that is the case.

Nuclear and chemical reaction can happen even when the energy is not sufficient to overcome the barrier by quantum mechanical tunneling. Tunneling is more probable for light particles. This effect actually plays a role in making reaction (2) more efficient than reaction (1): a deuteron ${}^{2}H$ can tunnel from one nitrogen nucleus to the other, so there is no need for the two nitrogen nuclei to come close and to overcome the first barrier.

Solution 4

We can estimate the ionization energy of N^{6+} using the Rydberg equation for hydrogen-like systems, given that it is also a one-electron system, but with a 7 times higher nuclear charge. The binding energy of the electron is:

$$
E_{bind} = -\frac{hc\mathcal{R}Z^2}{/} n^2, \tag{4}
$$

with the Rydberg constant $hc\mathcal{R} = 2.179 \times 10^{-18}$ J. This results in an ionization energy of 0.0007 MeV. Given that the ionization energies of the 6 other electrons is even smaller (given the smaller positive charge of the ion core), we can assume the nitrogen atoms are fully ionized at temperatures above 1 MeV.

Solution 5

For the collision of two like particles, the dipole moment $\vec{d} = q_1 * r_1 + q_2 * r_2 = q(r_1 + r_2)$ is proportional to the center of mass $m(r_1 + r_2)$, which is a constant of motion. Hence, the time derivative of the dipole moment is zero, meaning no power is radiated.

Electrons are the main radiators in an electron-ion collision, since the relative accelerations (second time derivetive of $\vec{r}$) are inversely proportional to the masses.

Solution 6

Let λ_1 and λ_2 be the wavelengths of the incident and scattered x rays, respectively, as shown in Figure 3-18. The corresponding momenta are

$$p_1 = \frac{E_1}{c} = \frac{hf_1}{c} = \frac{h}{\lambda_1}$$

and

$$p_2 = \frac{E_2}{c} = \frac{hf_2}{c} = \frac{h}{\lambda_2}$$

using $f\lambda = c$. Since Compton used the K_α line of molybdenum ($\lambda = 0.0711$ nm; see Figure 3-15*b*), the energy of the incident x ray (17.4 keV) is much greater than the binding energy of the valence electrons in the carbon-scattering block (about 11 eV); therefore, the carbon electrons can be considered to be free.

Conservation of momentum gives

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_e$$

or

$$\begin{aligned} p_e^2 &= p_1^2 + p_2^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\ &= p_1^2 + p_2^2 - 2p_1p_2\cos\theta \end{aligned} \qquad \textbf{3-26}$$

where $\boldsymbol{p}_e$ is the momentum of the electron after the collision and θ is the scattering angle of the photon, measured as shown in Figure 3-18. The energy of the electron before the collision is simply its rest energy $E_0 = mc^2$ (see Chapter 2). After the collision, the energy of the electron is $(E_0^2 + p_e^2c^2)^{1/2}$.

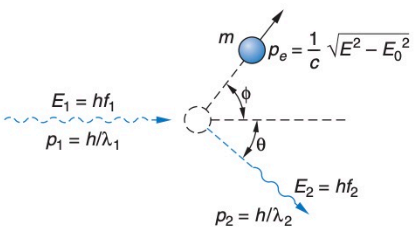


FIGURE 3-18 The scattering of x rays can be treated as a collision of a photon of initial momentum h/λ_1 and a free electron. Using conservation of momentum and energy, the momentum of the scattered photon h/λ_2 can be related to the initial momentum, the electron mass, and the scattering angle. The resulting Compton equation for the change in the wavelength of the x ray is Equation 3-25.

Conservation of energy gives

$$p_1c + E_0 = p_2c + (E_0^2 + p_e^2c^2)^{1/2}$$

Transposing the term p_2c and squaring, we obtain

$$E_0^2 + c^2(p_1 - p_2)^2 + 2cE_0(p_1 - p_2) = E_0^2 + p_e^2c^2$$

or

$$p_e^2 = p_1^2 + p_2^2 - 2p_1p_2 + \frac{2E_0(p_1 - p_2)}{c} \qquad \textbf{3-27}$$

Eliminating p_e^2 between Equations 3-26 and 3-27, we obtain

$$\frac{E_0(p_1 - p_2)}{c} = p_1p_2(1 - \cos\theta)$$

Multiplying each term by $hc/p_1p_2E_0$ and using $\lambda = h/p$, we obtain *Compton's equation*:

$$\lambda_2 - \lambda_1 = \frac{hc}{E_0}(1 - \cos\theta) = \frac{hc}{mc^2}(1 - \cos\theta)$$

or

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta) \qquad \textbf{3-25}$$

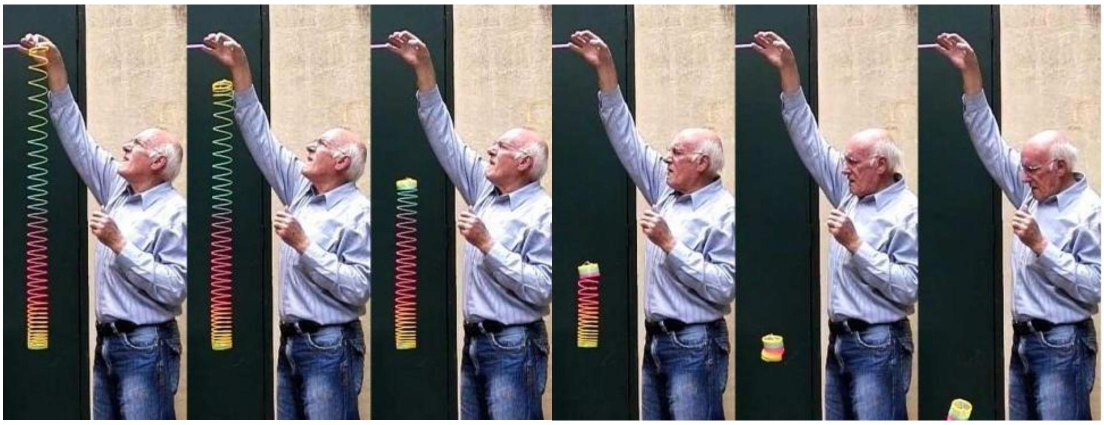
For $\theta = \pi$ is $\Delta\lambda$ at its maximum.

Solution 7

No, the rate for cooling through bremsstrahlung is always larger than the rate for releasing energy through fusion of two nitrogen nuclei. Hence, the fusion reaction cannot be sustained.

Falling Slinky (answers)

The physics of a falling slinky has been discussed among others by prof. Shimon Kolkowitz in http://large.stanford.edu/courses/2007/ph210/kolkowitz1/
and by Daniel Walsh in
http://danielwalsh.tumblr.com/post/11566016253/explaining-an-astonishing-slinky
but there are also many alternative descriptions available, like for instance the one presented below.



a) The pulling force in each part of the slinky is determined by its **relative local extension**, such that Hooke's law can be written as $F(x)=kdL/dx$, where the coordinate $x \in [0,1]$ denotes the position on the slinky. As this force has to lift the lower part of the slinky $F(x)=mgx= kdL/dx$. Integration of this equation yields the shape of the slinky $L(x)=[mg/(2k)]x^2$. The total length of the slinky is $L_0=mg/(2k)$ is needed for later reference.
b) When the top part of the slinky is falling, **the bottom part doesn't yet notice this because its local shape hasn't yet changed**. The acceleration of the top part is faster than that of a free-falling object, as this top part experiences the pulling force of the lower part.
c) In order to calculate how long it takes for the top of the slinky to reach the bottom of the slinky, you don't need to solve its full equation of motion. It is enough to **consider the motion of the center of mass of the slinky**, which is originally positioned at $L_0/3$ above the bottom of the slinky. This center of mass moves as any free falling object does and reaches the bottom at a time t_{fall} that obeys the equation $(L_0/3)=(g/2)(t_{fall})^2$. This yields $t_{fall}= \sqrt{(2L_0/3g)}$, which is a factor $\sqrt{3}$ shorter than the fall time $\sqrt{(2L_0/g)}$ of a point-like object falling over a distance L_0.
d) We can derive an equation for the distance $\Delta L(t)$ travelled by the top of the slinky at a time t after 'launch', up to moment when it reaches the bottom of the slinky, by combining the equation for the motion of the center of mass with the observation that a compression wave travels neatly from top to bottom through the slinky. When at a time t a fraction $y=1-x$ of the top of the slinky has collapsed, the position of the center of mass with respect to the bottom of the slinky can be written as $L_0/3-½gt^2=L_0(x^2-(2/3)x^3)$. This equation describes how the slinky contracts in time, but the solution $x(t)$ is far from trivial. Its time derivation yields the expression $dx/dt=-gt/[2L_0x(1-x)]$ and the real speed $v=-dL/dt=-2L_0x.dx/dt=gt/(1-x)$. As an alternative approach towards a solution, we can consider the acceleration of the contracted top section of the slinky, which as $dv/dt=g/(1-x)$. I would like to check whether these two approaches yield the same result. I would also like to proof that the rest of the slinky remains unaffected when the top part contracts, but this might be too difficult for now.

Charged particles around a black hole (with solutions)

In this exercise, the unit system c = G = 1 is used. All answers can be expressed in these units.

The spherically symmetric vacuum solution of the Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

is given by the famous Schwarzschild spacetime: a non-rotating black hole of mass *M*. If this spacetime is doused in a constant magnetic field $F^r{}_\varphi$ that is small enough to have negligible effect on the curvature of spacetime, equatorial and circular motion of a charged particle of mass *m* and charge *q* is given by:

$$\frac{M}{r^2}\left(1-\frac{2M}{r}\right)(u^t)^2 - r\left(1-\frac{2M}{r}\right)(u^\varphi)^2 = \frac{q}{m} F^r_\varphi u^\varphi$$

in here, $u^\mu = (u^t, u^r, u^\theta, u^\varphi)$ is the 4-velocity of the particle, measured in proper time τ. All these velocities are constant. The only non-zero component magnetic field *B* corresponding to $F^r{}_\varphi$ is perpendicular to the equatorial plane, using that in this geometry

$$|F^r_\varphi| = |B_\theta| = |B_z|\, r\left(1-\frac{2M}{r}\right)$$

a).
Explain, without calculations, for each of these velocities (u^t, u^r, u^θ, u^φ) why it is constant.

Solution:
Since the motion is in the equatorial plane and circular, θ = const, r = const, giving $u^\theta = 0$, $u^r = 0$. The orbital velocity is constant as the forces do not change along the orbit, making u^ϕ = constant. Finally, since the velocity respectively the distance are constant, the special relativistic time dilation respectively gravitational time dilation are necessarily constant, leading to $u^t = 0$.

This particle does not simply follow the usual (Newtonian) Kepler's Third Law, but a modified one based on the fact that it moves in a gravitational field and is subject to a magnetic field.

b).
Show that the modified version of Kepler's Third Law is given by

$$u^\varphi = \frac{-\frac{q}{m}F^r_\varphi \pm \sqrt{\left(\frac{q}{m}F^r_\varphi\right)^2 + \frac{4M}{R}\left(1-\frac{3M}{R}\right)}}{2R\left(1-\frac{3M}{R}\right)}$$

in which *R* is the radius of the circular orbit.

Solution:
From the line-element or the normalisation of 4-velocity, $g_{\mu\nu}u^\mu u^\nu = -1$, which for this situation (Schwarzschild and circular orbit, using results from a) results in $-(1-2M/R)(u^t)^2 + r^2 (u^\phi)^2 = -1$. Using the relation between u^t , u^ϕ stated in the exercise, a quadratic equation for the angular velocity can be obtained, which has the requested as its outcome.

In absence of a magnetic field, the smallest allowed circular orbit around a Schwarzschild black hole has radius $R = 3M$. From the result of exercise **b**, we see that circular motions exist around the black hole with smaller radii, provided the magnetic field is large enough. A magnetic dipole field $F^r{}_\varphi$ with magnetic dipole moment μ in a Schwarzschild spacetime, can be shown to be given by:

$$F^r_\varphi = \frac{\mu}{R^2}\left(1-\frac{2M}{R}\right)\left(h - R\frac{\partial h}{\partial R}\right) \quad \text{in which} \quad h(R) = \frac{3R^3}{8M^3}\left(\ln\left(1-\frac{2M}{R}\right)+\frac{2M}{R}+\frac{2M^2}{R^2}\right)$$

This results in some interesting regions R of allowed circular orbits. For a range of values $\mu > 0$, there exists a region $2M < R < 3M$ in which no orbits are allowed: a forbidden zone.
However, if μ is made big enough, circular orbits are allowed for all values R in between $2M < R < 3M$, and the forbidden zone disappears.

c).
From the expression in exercise **b**, calculate the formula for the non-zero magnetic field strength μ needed to make the forbidden zone disappear.

Solution:
The 'forbidden zone' is the result of the 4-velocity being a non-real expression, which is when the discriminant of the Kepler's Third Law above is positive. The point at which it flips from negative to positive is when the discriminant is zero. Substituting the expression for $F^r{}_\phi$ given above, the expression for the magnetic dipole moment can be found to be:

$$\mu^2 = -\frac{4M}{R}\left(1-\frac{3M}{R}\right)\left(\frac{m}{q}^2\right)\frac{R^4}{1-\frac{2M}{R}}\frac{1}{\left(h-\frac{\partial h}{\partial R}\right)^2}$$

While the particle is doing its orbiting, it will send out both electromagnetic radiation as gravitational waves, losing energy due to both in the process. In what follows, we will investigate the stability of these orbits under sending out of this energy.

To do so, we will need the energy loss due to electromagnetic radiation. Maxwell's electrodynamics teaches us that the power P_{elec} sent out by an accelerating particle in the non-curved background of Minkowski spacetime is given by

$$P_{\text{elec}} = \frac{q^2\gamma^4}{6\pi}\left(\vec{a}^2 + \gamma^2(\vec{v}\cdot\vec{a})^2\right)$$

in which γ is the special relativistic Lorentz-factor, v is the particle's orbital velocity, and a its corresponding acceleration, both measured in t.

d).
From Newton's second law $dp/dt = F$ in a Minkowski background and from the Minkowski line-element, show that in our current situation the following hold:

$$|\vec{a}| = \frac{q\,|\vec{v}|}{\gamma^2 m^2}|B_z| \qquad\qquad \vec{a}\cdot\vec{v} = 0$$

in which the inner product is the usual three-dimensional dot-product.

Solution:
The Minkowski line-element and/or the normalisation of 4-velocity $\eta_{\mu\nu}u^\mu u^\nu = -1$ can be differentiated with respect to proper time to result in $\eta_{\mu\nu}\, u^\mu\, du^\nu/d\tau = 0$. With the time-dilation being constant, what remains is $\eta_{ij}\, u^i\, du^j/d\tau = 0$, resulting what is requested.
From the Minkowski version of Newton's Second Law: the right hand side is $F = \gamma\, q\, (v \text{ cross } B_z)$. With the magnetic field only having a z-component, this force only points in x- and y-directions, $F_x = q\, v_y\, B_z$, $F_y = -q\, v_x\, B_z$. The left hand side is, using $p = \gamma\, m\, v$, equal to $\gamma^3 ma$. Setting equal and taking the magnitude on both sides results in the requested expression for the acceleration.

Using the result of exercise **d**, we can calculate the electromagnetic energy loss in a Minkowski background. The power sent out by gravitational waves for a mass m in a circular orbit around a mass *M* is, in a Minkowski background, given by the Peters-Mathews equation:

$$P_{GW} = \frac{32}{5}\,(M_c\omega)^{10/3} \qquad \text{in which} \qquad M_c = \frac{(mM)^{3/5}}{(m+M)^{1/5}}$$

Taking the ratio of P_{elec} and P_{GW}, we can now calculate which of the two radiations dominates the energy loss. However, the formulas for the powers studied in **c** and **d** work in a Minkowski background. In our current situation we need a Schwarzschild background.

e).
Explain how the ratio between electromagnetic power and gravitational wave power changes when we move from Minkowski background to Schwarzschild background. Note: no calculations necessary.

Solution:
The ratio of the electromagnetic radiation and gravitational wave energies here given is in the Minkowski background. When going to a Schwarzschild background, both types of radiation will experience redshift, the amount of which is the same for both types due to the Strong Equivalence Principle. This results in the ratio staying unchanged when going from Minkowski to Schwarzschild.

Supersymmetric quantum mechanics and exactly solvable models. Solution

(a) *Consider two Hamiltonians, $\hat{H}_1 = \hat{A}^\dagger \hat{A}$ and $\hat{H}_2 = \hat{A}\hat{A}^\dagger$, where*

$$\hat{A} = -\frac{d}{dx} + W(x) \tag{1}$$

with real $W(x)$. Show that $\hat{H}_1$ and $\hat{H}_2$ are Hermitian operators with non-negative eigenvalues.

Hermiticity:

$$\hat{H}_1^\dagger = \left(\hat{A}^\dagger \hat{A}\right)^\dagger = \hat{A}^\dagger \left(\hat{A}^\dagger\right)^\dagger = \hat{A}^\dagger \hat{A} = \hat{H}_1 \tag{2}$$

and similarly for $\hat{H}_2$.

Non-negativity of energy: Consider an expectation value of $\hat{H}_1$ in a state $|\psi\rangle$:

$$\langle\psi|\hat{H}_1|\psi\rangle = \int_{-\infty}^{+\infty} dx\, \psi^*(x)\hat{A}^\dagger \hat{A}\psi(x) = \int_{-\infty}^{+\infty} dx \left(\hat{A}\psi(x)\right)^* \left(\hat{A}\psi(x)\right) = \langle\phi|\phi\rangle, \tag{3}$$

where $|\phi\rangle = \hat{A}|\psi\rangle$. Since the norm of any physical state is non-negative, $\langle\psi|\hat{H}_1|\psi\rangle \geq 0$. If $|\psi\rangle$ is an eigenstate of $\hat{H}_1$ with the energy ε, then

$$\langle\psi|\hat{H}_1|\psi\rangle = \varepsilon \geq 0, \tag{4}$$

where we assumed that $|\psi\rangle$ is normalized by $\langle\psi|\psi\rangle = 1$. The proof for $\hat{H}_2$ is similar.

(b) *Prove that if $\hat{H}_1$ has an eigenfunction $\psi_1(x)$ with a nonzero energy ε, then there is an eigenfunction $\psi_2(x)$ of $\hat{H}_2$ with the same energy. Find a relation between $\psi_1(x)$ and $\psi_2(x)$.*

From $\hat{H}_1|\psi_1\rangle = \varepsilon|\psi_1\rangle$ we obtain

$$\hat{A}\hat{H}_1|\psi_1\rangle = \varepsilon\hat{A}|\psi_1\rangle. \tag{5}$$

Noting that

$$\hat{A}\hat{H}_1|\psi_1\rangle = \hat{A}\hat{A}^\dagger\hat{A}|\psi_1\rangle = \hat{H}_2\hat{A}|\psi_1\rangle, \tag{6}$$

we find

$$\hat{H}_2|\psi_2\rangle = \varepsilon|\psi_2\rangle, \tag{7}$$

where $|\psi_2\rangle = \hat{A}\hat{\psi}_1$. Similarly, for the eigenstate $|\psi_2\rangle$ of $\hat{H}_2$ with the energy ε, we have

$$\hat{A}^\dagger\hat{H}_2|\psi_2\rangle = \hat{A}^\dagger\hat{A}\hat{A}^\dagger|\psi_2\rangle = \hat{H}_1\hat{A}^\dagger|\psi_2\rangle = \varepsilon\hat{A}^\dagger|\psi_2\rangle, \tag{8}$$

so that $|\psi_1\rangle = \hat{A}^\dagger|\psi_2\rangle$ is the eigenstate of $\hat{H}_1$ with the same energy.

(c) *Prove that the zero-energy eigenfunction $\psi_1(x)$ of $\hat{H}_1$ (if it exists) satisfies*

$$\hat{A}\psi_1(x) = 0, \tag{9}$$

whereas the zero-energy eigenfunction $\psi_2(x)$ of $\hat{H}_2$ satisfies

$$\hat{A}^\dagger \psi_2(x) = 0. \tag{10}$$

The answer to question (b) does not hold for $\varepsilon = 0$. If $\hat{H}_1|\psi_1\rangle = 0$, then

$$\langle \psi_1|\hat{H}_1|\psi_1\rangle = \langle \phi|\phi\rangle = 0, \tag{11}$$

where $|\phi\rangle = \hat{A}|\psi_1\rangle$ [see Eq.(3)]. The norm of the state $|\phi\rangle$ is only 0, when $\phi(x) \equiv 0$. Hence, $|\psi_1\rangle$ satisfies Eq.(9). Similarly, from $\hat{H}_2|\psi_2\rangle = 0$, we obtain that the wave function $\hat{A}^\dagger \psi_2(x) = 0$ for all x.

It is easy to solve Eqs.(5) and (6) for arbitrary $W(x)$ and show that only one of the two Hamiltonians (or none) can have the zero-energy eigenstate.

To shorten the problem, you were not asked to verify these statements, but the solution of

$$\hat{A}\psi_1 = -\frac{d\psi_1}{dx} + W\psi_1 = 0 \tag{12}$$

is

$$\psi_1(x) \propto \exp\left(\int_0^x dx' W(x')\right) \tag{13}$$

and the solution of Eq.(10) is

$$\psi_2(x) \propto \exp\left(-\int_0^x dx' W(x')\right). \tag{14}$$

Clearly, $\psi_1(x)$ and $\psi_2(x)$ cannot be both normalizable.

(d) *Show that for $W(x) = N\tanh(x)$,*

$$V_1(x) = N^2 - \frac{N(N-1)}{\cosh^2 x} \quad \textit{and} \quad V_2(x) = N^2 - \frac{N(N+1)}{\cosh^2 x}. \tag{15}$$

First, we notice that

$$\hat{A}^\dagger = \frac{d}{dx} + W(x) \tag{16}$$

(the operator $\frac{d}{dx} = \frac{i}{\hbar}\hat{p}$, where $\hat{p}$ is the momentum operator, changes sign under Hermitian conjugation). Therefore,

$$\hat{H}_1 = -\frac{d^2}{dx^2} + W^2(x) + \frac{dW}{dx}(x) \equiv -\frac{d^2}{dx^2} + V_1(x),$$
$$\hat{H}_2 = -\frac{d^2}{dx^2} + W^2(x) - \frac{dW}{dx}(x) \equiv -\frac{d^2}{dx^2} + V_2(x).$$

The rest of the calculation is straightforward:

$$V_1(x) = \frac{dW}{dx} + W^2 = \frac{N}{\cosh^2 x} + N^2\frac{(\cosh^2 x - 1)}{\cosh^2 x} = N^2 - \frac{N(N-1)}{\cosh^2 x} \tag{17}$$

and similar for $V_2(x)$.

(e) *For $N = 1$, $V_1(x) = 1$. Find the ground state energy ε and the corresponding eigenstate $\psi_1(x)$ of $\hat{H}_1$. Find the eigenstate $\psi_2(x)$ of $\hat{H}_2$ with the same energy. Argue that $\psi_2(x)$ is not the ground state of $\hat{H}_2$.*

The eigenstates of $\hat{H}_1 = -\frac{d^2}{dx^2} + 1$ are plane waves, e^{ikx}, and the state with $k = 0$, $\psi_1(x) = 1$, has the lowest-energy state $\varepsilon = 1$. Then,

$$\psi_2(x) = \hat{A}\psi_1(x) = \tanh(x). \tag{18}$$

The ground state wavefunction for the potential $V_2(x) = 1 - \frac{2}{\cosh^2(x)}$ should be an even function of x and should have no zeros at finite x (oscillation theorem), so $\tanh(x)$ cannot be the ground state wavefunction.

(f) *Show that the ground state energy of $\hat{H}_2$ is 0 and find the corresponding wave function. Argue that $\hat{H}_2$ has no other bound states.*

Hint: Solve Eq.(10) for $W(x) = \tanh(x)$.

The solution of

$$\hat{A}^\dagger\psi(x) = \frac{d\psi}{dx} + \tanh(x)\psi = 0 \tag{19}$$

is

$$\psi(x) = A\exp\left(-\int_0^x dx'\frac{\sinh(x')}{\cosh(x')}\right) = A\exp\left(-\ln(\cosh x)\right) = \frac{A}{\cosh(x)}, \tag{20}$$

where $A = \frac{1}{\sqrt{2}}$ is the normalization constant and we used $\frac{d}{dx}\cosh(x) = \sinh(x)$. It is the wavefunction of bound state. The Hamitonian $\hat{H}_2$ has no other bound states with the energy ε: $0 < \varepsilon < 1$, because then $\hat{H}_1$ would have an eigenstate with the same energy, but the lowest energy in the spectrum of $\hat{H}_1$ is 1. The states with $\varepsilon \geq 1$ belong to the continuum spectrum for both Hamiltonians.

To summarize, the Hamiltonian

$$\hat{H} = -\frac{d^2}{dx^2} - \frac{2}{\cosh^2 x} \tag{21}$$

has one bound state with the energy $\varepsilon = -1$*, the wave function of which can be found analytically. Considering* $N = 2, 3, \ldots$*, one can find in a similar way the wave functions and energies of bound states of the Hamiltonian* $\hat{H} = -\frac{d^2}{dx^2} - \frac{N(N+1)}{\cosh^2 x}$*, but we stop here.*

The Hamiltonian $\hat{H}_1$, for $N = 2$, coincides with $\hat{H}_2$, for $N = 1$. Acting with the operator $\hat{A}$ on the bound state of $\hat{H}_1$ ($N = 2$), one obtains a bound state of the $\hat{H}_2$, for $N = 2$, which again cannot be its ground state because the $\hat{A}$ is odd under $x \to -x$. Hence, the new $\hat{H}_2$ should have a zero-energy state, etc etc.

Measurement of the B meson decay time distribution at the PEP-II collider

An e^+e^- collider is a collider that collides electrons (e^-) with positrons (e^+). If the beam energies are chosen such that the total energy is around 10 GeV (about 10 times the proton mass), several resonances can be seen (Fig. 1). These resonances are called the Upsilon (Υ) resonances. They are meta-stable states $b\bar{b}$ states, where b is the bottom (or 'beauty') quark.

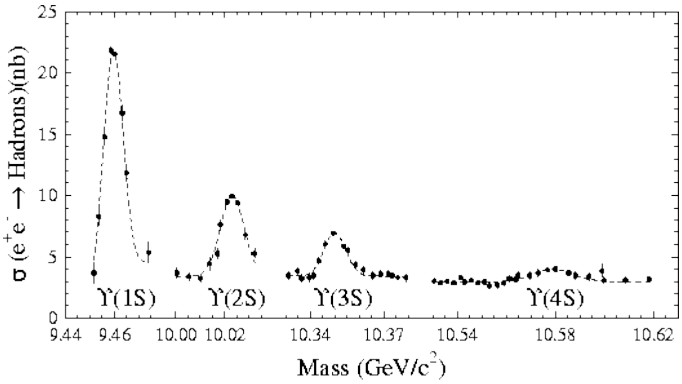


Figure 1: Particle production as a function of centre-of-momentum energy in the region of the Upsilon states as measured by the CUSB detector at Cornell in 1980.

Beauty mesons are mesons consisting of a beauty quark and a lighter anti-quark (or vice versa). The quark content of the four lightest beauty mesons is

$$B^0 = d\bar{b} \qquad \bar{B}^0 = \bar{d}b \qquad B^+ = u\bar{b} \qquad B^- = \bar{u}b$$

The rest mass of the B^0 and B^+ meson are almost identical, approximately $m_B = 5.279$ GeV$/c^2$. The $\Upsilon(4S)$ resonance at $M_\Upsilon = 10.580$ GeV$/c^2$ (the right-most bump in the figure) is just heavy enough for the decay into two beauty mesons: The majority of events at this collision energy is either $e^+e^- \to \Upsilon(4S) \to B^0\bar{B}^0$ or $e^+e^- \to \Upsilon(4S) \to B^+B^-$.

The decay of the upsilon to the two B mesons is a two-body decay: In the upsilon rest frame (also called the centre-of-momentum-system, or 'cms'), the two B mesons fly in opposite direction with the same momentum.

(a) Compute the momentum p_{cms} of a B meson in the upsilon rest frame. Express your answer in M_Υ and m_B.
(Here, and in the remainder of the exercise, you may choose to work in natural units, such that $c = 1$.)

$$\text{Answer:} \quad M_\Upsilon = 2\sqrt{m_B^2 + p^2/c^2} \Longrightarrow p = c\sqrt{M_\Upsilon^2/4 - m_B^2} \approx 0.34\ \text{GeV}/c$$

The PEP-II collider in Stanford is an e^+e^- collider tuned at the $\Upsilon(4S)$ resonance. PEP-II is an *asymmetric* collider: the positron beam has a lower energy than the electron beam, such that the two B mesons are boosted. The B mesons fly with almost the same velocity parallel to the electron beam.

(b) Given that the positron beam has an energy of $E_+ = 3.1$ GeV, compute the energy E_- of the electron beam. You may ignore the electron and positron mass, e.g. assume $p(e^+) = E_+/c$ and $p(e^-) = E_-/c$. Express the answer in terms of E_+ and M_Υ.

> Answer: $E_- = (M_\Upsilon c^2)^2/4E_+ \approx 9.03$ GeV.
> Ignore the electron/positron mass. Electron and positron travel in opposite directions along a common axis. Assume the electron to travel in the positive z direction. The total Upsilon momentum is then $p_{z,\Upsilon} = E_- - E_+$. The total energy is $E_\Upsilon = E_- + E_+$. Therefore, the length-squared of its four-vector $(E_- + E_+)^2 - (E_- - E_+)^2 = 4E_-E_+$. This needs to be equal to M_Υ^2.

(c) Compute the momentum p_{lab} of a B mesons in the laboratory frame *ignoring* its momentum in the centre-of-momentum frame: That is, assume that the B particle is at rest in the $\Upsilon(4S)$ rest frame. Express your answer in M_Υ, m_B, E_- and E_+.
Hint: Remember that $p = \gamma\beta mc$, where $\gamma\beta$ is the boost factor. Compute the boost factor $\gamma\beta$ for the upsilon. If you ignore the velocity of the B in the upsilon frame, the boost factor for the B mesons is identical to that for the upsilon.

> Answer: $p_B c \approx (E_- - E_+) m_B/m_\Upsilon \approx 2.96$ GeV.
> The more complicated answer will include a correction for the B momentum in the cms, which depends on the B polar angle. The average will then be the quadratic sum $\langle p_B \rangle = \sqrt{((E_- - E_+)m_B/m_\Upsilon)^2 + M_\Upsilon^2/4 - m_B^2}$ which is just a tiny bit larger.

The decay time-distribution of an unstable particle usually follows an exponential law

$$N(t) \;=\; N_0\, e^{-t/\tau} \tag{1}$$

where τ is the mean decay time. B mesons have an average lifetime τ_B of about 1.5 ps. Due to a phenomenon called CP-violation there exists decays for which the decay time distribution of $B^0 \to X$ is different from the decay time distribution of $\bar{B}^0 \to \bar{X}$. The aim of the experiment at Stanford is to measure this small difference. Therefore, it is important to measure the decay times very precisely.

The decay length L_{lab} is the distance between the point of decay and the point of production of the B meson in the laboratory frame. The decay time in the laboratory is computed by dividing the decay length by the measured velocity:

$$t_{\text{lab}} \;=\; \frac{L_{\text{lab}}}{v_{\text{lab}}} \tag{2}$$

(d) Show that the *proper* decay time t (e.g. the decay time in the rest frame of the B meson) can be computed as

$$t \;=\; \frac{L\, m_B}{p} \tag{3}$$

where m is the B meson rest mass, and p and L are respectively the B momentum and decay length in the *laboratory frame*, or any other frame in which the B meson is not at rest.

> Answer: Denote the velocity of the B meson with $v_{\mathrm{lab}} = \beta c$. This can be expressed in terms of the momentum with $p_{\mathrm{lab}} = \gamma m_B v_{\mathrm{lab}}$. The decay time in the lab t_{lab} is related to the decay time t in the rest frame by $t_{\mathrm{lab}} = \gamma t$. Therefore, we have: $t = t_{\mathrm{lab}}/\gamma = L_{\mathrm{lab}}/(\gamma v_{\mathrm{lab}}) = L_{\mathrm{lab}}/(\gamma\beta c) = mL_{\mathrm{lab}}/(\gamma\beta mc) = mL_{\mathrm{lab}}/p_{\mathrm{lab}}$

(e) Compute the average B meson decaylength (the distance a B meson travels before it decays) in the cms frame. Express the result in the average proper time τ_B, m_B and your answer to exercise *a*.

> Answer: $L_{\mathrm{cms}} = \tau\, p_{B,\mathrm{cms}} \,/\, m_B \approx 0.029$ mm.
> (The students are not asked to give the numerial answer at this stage, but they'll need it to answer the exercise (g).)

(f) Compute the average B meson decaylength in the laboratory frame, ignoring the velocity of the B meson in the e^+e^- rest frame. Express the result in τ_B, m_B and your answer to exercise *c*.

> Answer: $L_{\mathrm{lab}} = \tau\, p_{B,\mathrm{lab}} \,/\, m_B \approx 0.25$ mm.

It is technologically easier to build a symmetric-energy collider (e.g. with $E_+ = E_-$) than an asymmetric-energy collider. Yet, it was chosen to use the latter strategy for the PEP-II collider.

(g) The decay time resolution is determined by the decay length resolution. The latter is limited by technology: Typical particle detectors can reach a precision of about $\sigma(L) = 100\ \mu m$. Explain why the PEP-II collider was built as an asymmetric collider.

> Answer: The decay time resolution is essentially $\sigma(t) = \sigma(L)m_B/p$: For a given decay length resolution, the decay time resolution decreases with the momentum (or velocity). Therefore, the larger the momentum in the lab frame, the more precise one can measure decay times. To understand how much of a difference it makes, we compute the average decay length in the cms frame and in the lab frame:

$$
\begin{aligned}
L_{\mathrm{cms}} &= \tau\,\frac{p_{B,\mathrm{cms}}}{m_B} = \tau\, c\,\frac{p_{B,\mathrm{cms}}\, c}{m_B\, c^2} \\
&= \tau\, c\sqrt{\frac{m_\Upsilon^2}{4m_B^2} - 1} \\
&\approx (1.5\ \mathrm{ps})(3\cdot 10^8\ \mathrm{m/s})\sqrt{\left(\frac{10.58}{2\cdot 5.28}\right)^2 - 1} \approx 0.029mm
\end{aligned}
$$

$$
\begin{aligned}
L_{\rm lab} &= \tau\, c\, \frac{p_{B,lab}\, c}{m_B\, c^2} \\
&= \tau\, c\, \frac{1}{m_B c^2} (E_- - E_+) \frac{m_B}{m_\Upsilon} \;=\; \tau\, c\, \frac{E_- - E_+}{m_\Upsilon c^2} \\
&\approx (1.5 \text{ ps})(3 \cdot 10^8 \text{ m/s}) \frac{9.0 - 3.1}{10.58} \approx 0.25 mm
\end{aligned}
$$

(At this stage it is important to reintroduce the eventually omitted factors c: We do that by expressing mass an momentum in terms of mc^2 and pc respectively.) In a symmetric collider the average decay length of the B is about 30 micron, which is smaller than the detection resolution. In an asymmetric collider as PEP-II, the decay length is larger than the resolution, thanks to the 'boost'. This allows to measure the decay time with sufficient accuracy.

Vraag

Lichtabsorptie door een photosysteem gevolgd door ladingsscheiding.

Een fotosysteem bestaat uit een honderdtal pigmenten die met elkaar gekoppeld zijn, zodat ze na absorptie de excitatie energie zeer snel aan elkaar kunnen overdragen.
6% van de pigmenten (het reactiecentrum) is in staat om de excitatie door ladingsscheiding irreversibel om te zetten in een chemisch andere toestand.
de andere 94% van de pigmenten wordt antenne genoemd.
op deze manier wordt dus licht omgezet in chemische energie, die verder gebruikt wordt om CO_2 vast te leggen.
neem aan dat alle pigmenten een gelijke energie hebben.

1. hoeveel procent van de tijd is een excitatie dan in het reactiecentrum?
2. hoeveel procent van de tijd is een excitatie dan in de antenne?
3. neem aan dat een geëxciteerde toestand een natuurlijke levensduur heeft van 1 nanoseconde. neem aan dat de snelheid van ladingsscheiding overeenkomt met een levensduur van 1 picoseconde. De vervalsnelheid is het omgekeerde van de levensduur. Wat zijn de bijbehorende vervalsnelheden?
4. wat is dan het rendement van dit fotosysteem, maw hoeveel procent van de geabsorbeerde fotonen wordt door ladingsscheiding omgezet in een chemisch andere toestand? Neem aan dat de snelheden van energieoverdracht tussen antenne en reactiecentrum veel groter zijn dan de natuurlijke vervalsnelheid, en daarom buiten beschouwing kunnen blijven.
5. hoe zorgt de natuur dat de ladingsscheiding irreversibel is?

Antwoordmodel

Lichtabsorptie door een photosysteem gevolgd door ladingsscheiding.

Een fotosysteem bestaat uit een honderdtal pigmenten die met elkaar gekoppeld zijn, zodat ze na absorptie de excitatie energie zeer snel aan elkaar kunnen overdragen.
6% van de pigmenten (het reactiecentrum) is in staat om de excitatie door ladingsscheiding irreversibel om te zetten in een chemisch andere toestand.
de andere 94% van de pigmenten wordt antenne genoemd.
op deze manier wordt dus licht omgezet in chemische energie, die verder gebruikt wordt om CO_2 vast te leggen.
neem aan dat alle pigmenten een gelijke energie hebben.

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   6%
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   $1ns^{-1}$ en $1ps^{-1}$
4. wat is dan het rendement van dit fotosysteem, m.a.w. hoeveel procent van de geabsorbeerde fotonen wordt door ladingsscheiding omgezet in een chemisch andere toestand? Neem aan dat de snelheden van energieoverdracht tussen antenne en reactiecentrum veel groter zijn dan de natuurlijke vervalsnelheid, en daarom buiten beschouwing kunnen blijven.

   $6\%*1ps^{-1}/(6\%*1ps^{-1}+94\%*1ns^{-1})=60/(60+0.94)=98.46\%$

5. hoe zorgt de natuur dat de ladingsscheiding irreversibel is?

   Doordat de ladingsgescheiden toestand een veel lagere (Gibbs vrije) energie heeft is de ladingsscheiding praktisch irreversibel.