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# PION 2020

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Natuurkunde Olympiade  
13 november



**Problem set**

# Nederlandse Natuurkundige Vereniging

## Voordelen van het NNV-lidmaatschap:

- Maandelijks het Nederlands Tijdschrift voor Natuurkunde in de bus
- Zeer aantrekkelijke ledenkorting op het jaarlijkse evenement FYSICA
- Optie op gratis lidmaatschap van de NNV-secties
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*Tweedejaarsstudenten kunnen een jaar gratis lid worden. Alle andere studenten krijgen een flinke korting op de contributie.*

De Nederlandse Natuurkundige Vereniging bestaat al sinds 1921 en is dé vereniging voor natuurkundigen in Nederland. De NNV is voor alle fysici: studenten, fysici werkzaam in het bedrijfsleven, onderwijs, academia...  
Immers:  
Eenmaal fysicus, altijd fysicus!

Gezicht van de natuurkunde



## Preface

Dear contestants,

On the computer screen in front of you, you see the problem set of the PIONline 2020. After hard work and a lot of uncertainty the committee is proud to present this to you through this new online format. The Science Park in Amsterdam houses many different research organisations. Every part of physics is being studied and we took full benefit of this. The result is great variety in the exercises you are about to make! Astrophysics, Quantum Computation, Biophysics and several other hot topics are included in the problem set. Before you start, we would like you to carefully read the rules, which are different from other years and emphasize that this year is based on *academic honesty*. We know that you, *an academic student on the highest level*, can be trusted with this responsibility. We would like to thank the professors that worked hard on these exercises and the sponsors that made this contest possible. Finally we wish all participating teams good luck! *Do not go gentle into that good night, rage rage against the dying of the light.*

Stay healthy,

PION committee 2020

*Natuurwetenschappelijke Studievereniging Amsterdam (NSA)*

*From left to right*

SIEM FENNE [*Treasurer*]

THOMAS GOLDMAN [*Chair/secretary*]

QUINTEN COELHO [*Chair/secretary*]

DUNCAN PELAN

ADRIAN RUESINK

CHRISTIAAN VAN ASPEREN



## Rules

Rules and remarks about the competition.

### General rules

- It is not allowed to communicate (about the problems) by any means with anyone other than team members.
- It is allowed to use a scientific, graphical calculator (Ti-83, Ti-84 or a similar model).
- Mobile phones are tolerated, but can only be used for emergencies, during the duration of the competition.
- You are allowed to eat, drink and use the bathroom during the competition.

### Online edition rules

- The PIONline is largely based on academic honesty. We will not be able to completely monitor everything you do and therefore we decided to simply trust you. Below we clarify what we consider fair and what not.
- You are allowed to use the internet in order to search general formulas, identities and constants. Examples are:
  - Formulas that would be in a BiNaS
  - Maxwell equations
  - Trigonometric identities.
  - The proton mass.
- You are allowed to use a computer for doing calculations. For example the Windows calculator app, Wolfram Alpha and Jupyter notebook (Python) can be used to simply fill in equations with numbers.
- You are **not** allowed to use the internet in order to search for solutions or examples of problems that can be considered as reasonably similar to a PION exercise or part of a PION exercise. Examples could be:
  - Derivation of the speed of light using the Maxwell Equations.
  - When you are asked to sketch a plot of a certain result, you are not allowed to search for the plot or the curve of the plot.
  - Methods for solving differential equations
- You are **not** allowed to use a computer to solve mathematical problems like integrals and differential equations using software like Wolfram Alpha or Mathematica.

## Handing in your work

- After the 3 hours for the test, the participants are given 30 extra minutes to hand in the exercises. Prior to the start of the contest, the *exact* deadline time will be communicated.
- If your team encounters problems with handing in your work, please let the committee know as soon as possible! If nothing is heard before the deadline and you hand in nothing, you will be disqualified.
- Make sure your team name is written on every page (front and back)! Write on at least one page all the names in the team and their individual email addresses. Number your pages [1/10], [2/10], ... , [10/10].
- Your work should be handed in as a single .pdf. You are allowed to use L<sup>A</sup>T<sub>E</sub>X to create the .pdf file. When you decide to write your solutions out by hand, use a scanner or an app that scans a picture. A WebApp we recommend for merging pictures or .pdf files into a single .pdf file can be found here: <https://smallpdf.com/merge-pdf>
- The following methods are allowed to send the .pdf to the committee:
  - Mail to pion@nsaweb.nl
  - Mail to pion@nsaweb.nl using <https://wetransfer.com/> (When the file is too large to be directly mailed)
  - Through the zoom chat in the private breakout room
- **When you are finished and you send us your solutions, please call us to your private breakout room so we can verify that we received your work!**

## The winner!

- There will be a top 3 of teams that will be awarded prizes!
- At least one team in the top 3 is a bachelor team.  
*Example: if the 3 teams with the highest points are master teams, the third place will go to the bachelor team with the most points.*
- Two teams will be awarded a place in the *online* version of the PLANCKS which will be held on 11-13 December. At least one of those teams is the bachelor team with the most points. The other team is either the PION first place (if it is a master team) or the PION second place (if a bachelor team is first place).

## Contents

<b>1</b>	<b>The Power of Accretion</b>	<b>5</b>
<b>2</b>	<b>Higgs Mechanism in Electrodynamics</b>	<b>7</b>
<b>3</b>	<b>QUESTION M (Microtubule Length Control)</b>	<b>9</b>
<b>4</b>	<b>Gravitoelectromagnetism and AMO Physics</b>	<b>11</b>
<b>5</b>	<b>Nature's Near-Perfect Clocks</b>	<b>13</b>
<b>6</b>	<b>The Kapitza Pendulum</b>	<b>15</b>
<b>7</b>	<b>Topological Mechanical Metamaterials</b>	<b>17</b>
<b>8</b>	<b>Spin Dynamics Due To Mobile Electron</b>	<b>20</b>

## 1 The Power of Accretion

*Prof. dr. S.B. (Sera) Markoff*

*Anton Pannekoek Institute of Astronomy - University of Amsterdam*

*10 points*

Compact objects like black holes are often fed by gravitationally captured gas. In this problem, you will calculate the consequences of accretion with the aid of a few simple assumptions. Use cylindrical coordinates  $(R, z, \phi)$  throughout.

[A] (1 pt) Consider hydrogen gas captured at a very large distance, in orbit around a maximally rotating 10 solar mass black hole. Assume it eventually reaches the event horizon with radius  $R_{\text{hor}} = GM_{\text{BH}}/c^2$ , where  $M_{\text{BH}}$  is the mass, and that the energy gained on infall is converted into internal heat. What is the average temperature of the gas?

[B] (1 pt) If the gas is fully ionised it will form a plasma. Consider plasma near the black hole with velocity  $\vec{V}$ , electric field  $\vec{E}$  and magnetic field  $\vec{B}$  (all quantities are measured in the lab frame). In ideal magnetohydrodynamics (MHD), the plasma is assumed to be highly conducting (conductivity  $\sigma \rightarrow \infty$ ). Prove that the Lorentz force on a charged particle vanishes in the comoving frame. Using the result of your proof, find the relationship between  $\vec{E}$ ,  $\vec{V}$ , and  $\vec{B}$  in the lab frame. [*Hint: What is the force in the lab frame?*]

[C] (1 pt) Around black holes, magnetic fields can actually dynamically dominate the flow. Show that under the additional assumptions of axisymmetry ( $\partial/\partial\phi \rightarrow 0$ ) and time-independence ( $\partial/\partial t \rightarrow 0$ ),  $\vec{E}_\phi$  vanishes. In MHD it is convenient to decompose vectors in the form:  $\vec{B} = B_p \hat{p} + B_\phi \hat{\phi}$  where the  $B_p$  is called the poloidal component of the magnetic field (in the  $R - z$  plane) and  $B_\phi$  is the toroidal component. Now, using the decomposed vectors and  $E_\phi = 0$ , prove that  $\vec{V}_p = \kappa \vec{B}_p$  where  $\kappa$  is a proportionality constant.

Black hole systems (and other compact objects) are known to launch magnetised jets of plasma along their poles. How?

[D] (2 pts) Using conservation laws, we can show that  $V_\phi - \kappa B_\phi = R\Omega$  (where  $\kappa$  is the same proportionality constant for part (c) and  $\Omega$  is the local angular speed) and use it for a simplified expression of the electric field. Now compute the Poynting flux from the plasma using the expressions for the electric and magnetic fields you obtained in the previous part.

[E] (3 pts) Calculate the electromagnetic power transported by the Poynting flux in the poloidal direction passing through an area of arbitrary radius  $R$ . Use the fact that angular momentum conservation provides a relation between the poloidal and toroidal components of the magnetic field:  $B_\phi = \Omega R B_p / c$ .

[F] (2 pts) The upper limit of this electromagnetic power will be when all of the gravitational potential energy released goes into the magnetic energy. For this limiting value, calculate the strength of the poloidal magnetic field ( $B_p$ ) rotating with the Keplerian angular velocity ( $\Omega_K = \sqrt{GM_{\text{BH}}/R^3}$ ) over the event horizon area of a maximally rotating 10 solar mass black hole. Assume a mass accretion rate  $\dot{M} = 2.2 \times 10^{-8}$  solar masses per year (i.e., just a tiny fraction of captured mass from a companion star).

For comparison, the average magnetic field of our sun is of the order of 1-10 Gauss. So, you can see just how strong the magnetic fields are in the vicinity of black holes! Gravity ultimately powers electromagnetic fields that channel accretion power into, and accelerate, outflows and jets. This is why black holes and other compact objects like pulsars are able to launch powerful outflows!!



UNIVERSITEIT VAN AMSTERDAM



## 2 Higgs Mechanism in Electrodynamics

*Prof. dr. E.L.M.P. (Eric) Laenen*  
*Institute for Theoretical Physics - University of Amsterdam*  
*Dr. A. (Anne) de Visser*  
*Van der Waals-Zeeman Instituut - University of Amsterdam*  
 10 points

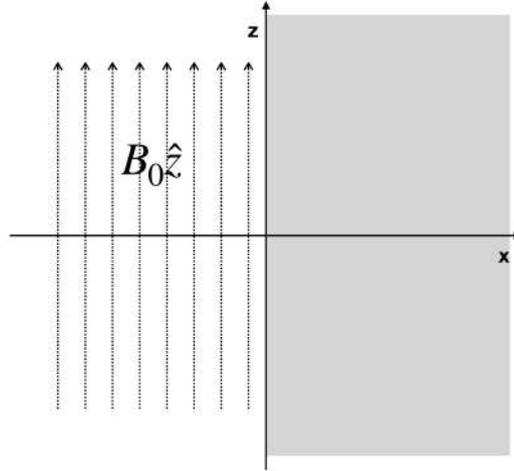


Figure 1

The Higgs mechanism of particle physics explains why certain particles have a mass rather than being massless like the photon propagating in vacuum. In this problem we shall see that such a situation already occurs in electrodynamics, for a magnetic field inside a BCS superconductor. In a BCS superconductor bound electron pairs (Cooper pairs) form a Bose condensate, which can act as a superconducting current.

Consider the setup of figure 1, in which there is a constant magnetic field pointing in the  $z$  direction outside the superconducting material. The material spans the entire region  $x \geq 0$ . A priori we know nothing about the magnetic field inside the material.

It is a special property of superconductors that the current density  $\mathbf{j}$  inside is related to the vector potential in the material as

$$\mathbf{j} = -m_A^2 \mathbf{A}, \quad (1)$$

where  $m_A$  is a constant. In a more detailed analysis it turns out that  $m_A = \mu_0 n_s e^2 / m$  with  $n_s$  the number density of Cooper pairs,  $e, m$  the electron charge and mass, but that is of no relevance here. Note that for convenience we rescaled the current density  $\mathbf{j} \equiv \mu_0 \mathbf{J}$ .

[A] (2 pts) Show that from (1) one can derive the second order differential equation

$$(\nabla^2 - m_A^2) \mathbf{B} = 0. \quad (2)$$

[B] (2 pts) Show that  $B_x = B_y = 0$  inside the material.

[C] (1 pts) Show that the general solution to the result of (2) can be written as  $B_z(x) = C_1 e^{\frac{x}{d}} + C_2 e^{-\frac{x}{d}}$ . How does  $d$  depend on  $m_A$ ? How does this solution change if we add  $\nabla \lambda$  to  $\mathbf{A}$  in (1), in which  $\lambda$  is an arbitrary function?

[D] (1 pts) Why must  $C_1$  vanish? Use boundary conditions to fix  $C_2$ .

[E] (2 pts) Check that your solution is consistent. I.e. given your result for the magnetic field, calculate the current density  $\mathbf{j}$ , use (1) to obtain the vector potential  $\mathbf{A}$  and consequently calculate the magnetic field  $\mathbf{B}$ .

[F] (2 pts) Sketch the magnetic field for  $x = -5d$  to  $x = 5d$ . Indicate the characteristic attenuation length  $x = d$  in the plot.

The result of our analysis is that magnetic fields are absent inside a superconductor, except in a very thin boundary layer. This phenomenon is known as the Meissner effect. The parameter  $m_A$  functions as an effective photon mass. Forces mediated by massive particles are in general short-range, hence the Meissner effect can be understood as the photon acquiring a mass inside the superconductor.

# Stichting Physica

### 3 QUESTION M (Microtubule Length Control)

Prof. dr. ir. E.J.G. (Erwin) Peterman  
 LaserLab - Vrije Universiteit  
 10 points

Just like our whole body, also our cells have an internal skeleton, the cytoskeleton, which gives the cell shape and mechanical stability. A major component of the cytoskeleton are microtubules, stiff polymers of the protein dimer tubulin, with the shape of a hollow pipe (25 nm diameter) that can be tens of micrometers long. Microtubules have a unique direction, signified by the plus end and minus end (see figure, panel A). In our cells, microtubules continuously grow and shrink by polymerisation and depolymerisation from the ends, and this process is highly regulated. One mechanism that has been proposed for microtubule length control involves the kinesin motor protein Kip3. Kip3 can bind to a microtubule, randomly along its length, and then walk in one direction, towards the microtubule plus end. Once it reaches the end, it can depolymerize the microtubule plus end by chipping off the terminal tubulin dimer as depicted in the figure below (panel A).

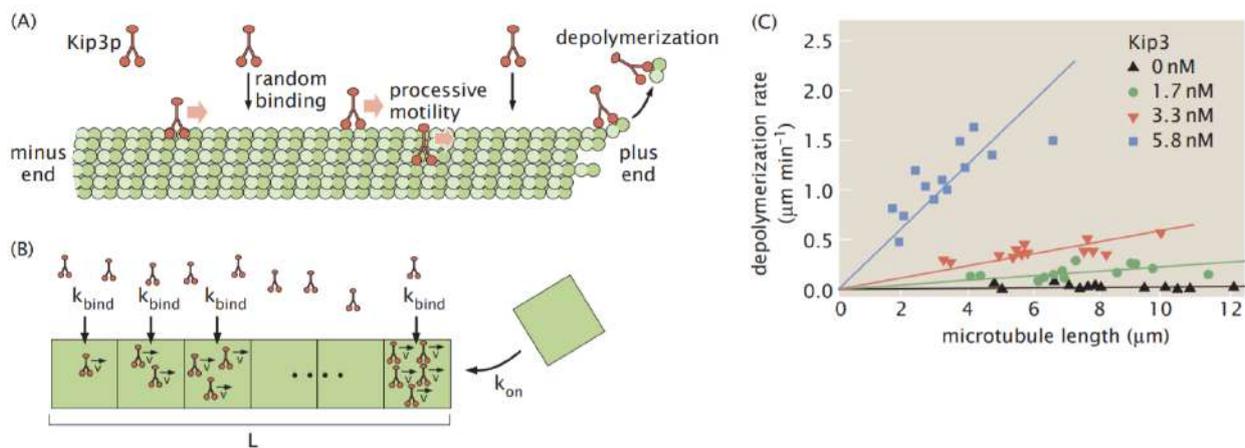


Figure 2: Antenna mechanism of microtubule length control. (A) Schematic of antenna mechanism of microtubule length control. (B) Toy model of the antenna mechanism. Kip3 molecules free in solution associate with microtubule monomers depicted as blocks, and then proceed toward the plus end moving at a constant speed  $v$ . Once at the end they detach from the microtubule taking a block/monomer with them. This Kip3 dependent depolymerization is counteracted by monomer addition at the plus end. (C) Data showing the length dependent depolymerization rate for different concentrations of Kip3. (A, adapted from Varga et al., Cell 138:1174, 2009; B, adapted from Varga et al., Nature Cell Biology 8:957, 2006)

In this problem, you will examine a toy model of this mechanism and show that it leads to a steady-state length for microtubules (i.e. microtubule length does not change with time). In the model (depicted in panel B of the figure), a microtubule is represented as a 1-dimensional lattice with length  $L$ , with each lattice site representing a monomer of the microtubule polymer. The model assumes that Kip3 motors bind from solution to any lattice site on the microtubule with rate  $k_{bind}$ , and then proceed to step toward the plus end of the microtubule (right in the figure) with rate  $k_{step}$ . Only when motors reach the end of a microtubule, Kip3 will fall off taking a lattice site along with them. This microtubule depolymerization process competes with monomer attachment at the plus end with rate  $k_{on}$ .

[A] (4 pts) Set up the rate equation for how  $n_i(t)$ , the number of Kip3 motors bound to the  $i^{th}$  monomer in the microtubule at time  $t$  changes with time. Here  $i$  ranges from  $1, 2, \dots, N$ , with  $N = \frac{L}{a}$  the number of monomers and  $a$  the length of a single monomer. Show that this rate equation predicts that in steady state (i.e. no changes in time) the number of Kip3 molecules

per monomer increases linearly with position toward the plus end as:  $n_i = n_{i-1} + \frac{k_{bind}}{k_{step}}$ . This has indeed been observed in the experiments of Varga et al. (2006). Forget for now what happens at the plus end.

- [B] (3 pts) Compute the rate of monomer detachment at the plus end in steady state assuming that it is equal to the flux of Kip3 molecules out of the terminal monomer. Compare your result with the data shown in the figure, which shows that, at a given concentration, the depolymerisation rate scales linear with polymer length.
- [C] (3 pts) Write down the rate equation for  $\frac{dL(t)}{dt}$ , the change in filament length with time, by considering polymerization at the plus end and the length-dependent depolymerization, which we derived in the previous part. Determine the steadystate length at 3nM Kip3 concentration ( $a \cdot k_{on} = 1 \mu\text{m} / \text{minute}$  and the Kip3 binding rate ( $k_{bind}$ ) is  $0.05 \text{ minute}^{-1}$ ).

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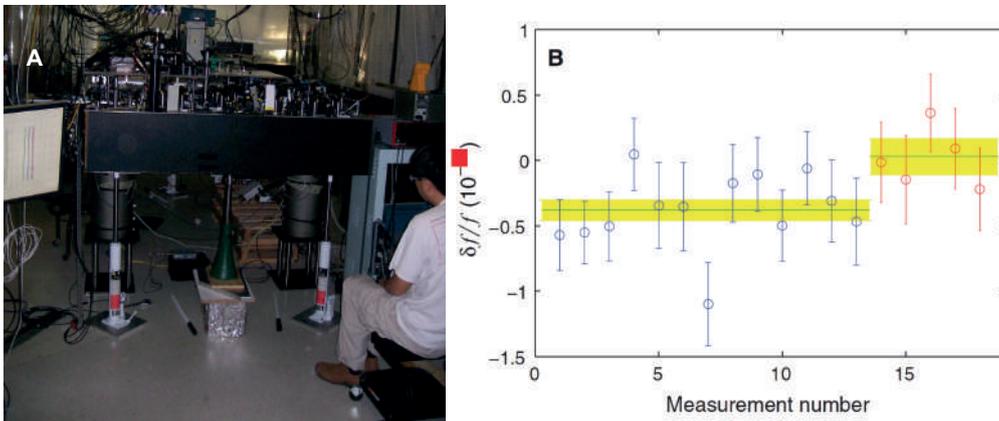


randstad

## 4 Gravitoelectromagnetism and AMO Physics

*Prof. dr. H.L. Bethlem and W. van der Meer, M.Sc.*  
*Quantum Metrology and Laser Applications – Vrije Universiteit Amsterdam*  
*10 points*

Nature has the convenient habit of using only a limited number of mathematical equations. There is a strong resemblance between Newton's law, that describes the force on a mass in a gravitational field, and Coulomb's law, that describes the force on a charge in an electric field. In the late 19th century, triggered by this resemblance, physicists speculated that the gravitational field must also have a 'magnetic' component, i.e., a velocity dependent force. A set of equations was put forward (see formula sheet). Later, after Einstein's seminal work, It was shown that under certain conditions, gravitoelectromagnetism is a good approximation to general relativity. This allows one to use his or her knowledge of electromagnetism to calculate the effects of general relativity on atomic and molecular spectra. In this problem, we will calculate the energy difference between clockwise and anti-clockwise spinning hydrogen molecules at the surface of the earth as a result of frame-dragging. To warm up, we will start with the aluminum ion clock.



**A:** Photo of the  $\text{Al}^+$  clock. **B:** Fractional uncertainty as a function of measurement number.

The group of Dave Wineland (Winner of the 2012 Nobel Prize in Physics), performed a famous experiment in which they lifted up the optical table on which the vacuum chamber is mounted that holds the aluminum ions by a distance of 33 cm, see panel A of the figure above. As a result of this change in height, the clock-transition in  $\text{Al}^+$  near 1.121 PHz changed by a measurable fraction, see panel B of the figure above (Ref. Chou et al. *Science* **329**, 1630 (2010)).

**Please note that there are some useful formulas on the next page!**

- [A]** (3 pts) Calculate the fractional change,  $\delta f/f$ , of the transition frequency as a result of the changing gravitational potential (as the transition frequency can be used as clock, this is the same as calculating how much faster time runs when you increase the gravitational potential!).

We will now turn to frame dragging.

- [B]** (4 pts) Calculate the gravitomagnetic field at the North Pole of the earth (at the surface). Feel free to simplify the problem by any means you see fit (anticipating that the energy difference will be too small to measure, an order of magnitude estimate suffices).
- [C]** (2 pts) Calculate the gravitomagnetic dipole moment of a hydrogen molecule in its first excited rotational state. Assume that the two protons that make up the hydrogen molecules are separated by 0.1 nm, and that the rotational frequency in this state is 3 THz.

[D] (1 pts) Use your answers from b and c to calculate the energy difference between clockwise and anti-clockwise spinning hydrogen molecules at the north pole. Calculate the fractional frequency difference.

**Some useful formulas for question 4**

Gravitational potential of point source with mass M:

$$\Phi_g = -\frac{GM}{r}$$

Gravitoelectromagnetic Coulomb law, Biot-Savart law, with  $\rho_g$  being the mass density, and  $\vec{J}_g = \rho_g \vec{v}$  the mass current.

$$\begin{aligned}\vec{E}_g &= G \int_{\text{volume}} \frac{\rho_g}{r^2} \hat{r} dV \\ \vec{B}_g &= \frac{G}{c^2} \int_{\text{volume}} \frac{\vec{J}_g \times \hat{r}}{r^2} dV\end{aligned}$$

Gravitoelectromagnetic Maxwell equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}_g &= -4\pi G \rho_g \\ \vec{\nabla} \cdot \vec{B}_g &= 0 \\ \vec{\nabla} \times \vec{E}_g &= -\frac{\partial \vec{B}_g}{\partial t} \\ \vec{\nabla} \times \vec{B}_g &= -\frac{4\pi G}{c^2} \vec{J}_g + \frac{1}{c^2} \frac{\partial \vec{E}_g}{\partial t}\end{aligned}$$

Gravitoelectromagnetic Lorentz force on testmass with mass,  $m$ :

$$\vec{F}_g = m \left( \vec{E}_g + \vec{v} \times 4\vec{B}_g \right)$$

## 5 Nature's Near-Perfect Clocks

*Dr. J.W.T. (Jason) Hessels*

*High Energy Astrophysics & Radio Astronomy - UvA & ASTRON*

*10 points*

A ‘pulsar’ is a rapidly rotating, highly magnetised neutron star. Some pulsars emit regular radio pulses at the rotational period ( $P_{\text{spin}}$ ) of the neutron star (NB: the rotational rate  $\nu_{\text{spin}} = 1/P_{\text{spin}}$ ). In the simple toy model of a pulsar (Figure 3), the neutron star has a dipole magnetic field and emits beams of coherent radio waves from above the magnetic polar caps. These rotate along with the neutron star, and a distant observer can view the sweeping beam as a series of highly periodic pulses occurring at a rate of  $\nu_{\text{spin}}$ .

The periodic pulses from a pulsar are like the ticks of a clock. We can record the times at which these pulses arrive at Earth. These ‘times of arrival’ (TOAs) depend on the pulsar’s rotational rate, as well as other effects like the rate at which the pulsar is ‘spinning down’ ( $\dot{\nu}_{\text{spin}}$ ) and its changing distance as the Earth orbits the Sun. If we can account for all of these effects, then we can create a ‘timing model’ and predict when the pulsar’s pulses will arrive at Earth. The actual observed TOAs, when compared to the model, allow us to refine these parameters. The difference between measurements and model is called the ‘residuals’.

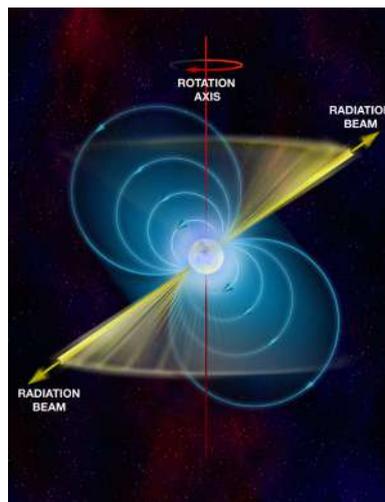


Figure 3: Toy model of a radio pulsar. The neutron star (centre) has a dipole magnetic field (blue lines) whose axis is at some angle with respect to the rotational axis of the star. Beams of coherent radio waves (yellow) are produced above the magnetic polar caps and rotate along with the neutron star. Credit: B. Saxton, NRAO/AUI/NSF.

- [A] (2 pts) The fastest-spinning pulsar known, which is named PSR J1748–2446ad, spins at  $\nu_{\text{spin}} = 716$  Hz. Can you use this fact to constrain the radius of the neutron star? Assume it has a mass  $1.4 \times$  that of the Sun, i.e.  $2.784 \times 10^{30}$  kg.
- [B] (4 pts) Consider Figure 4. This is showing the residuals of times of arrival (TOAs) compared to the timing model when a) all relevant physical parameters are well modeled with accurate parameter values and b) when the spin-down rate of the pulsar ( $\dot{P}_{\text{spin}}$ ) is poorly modeled. Using the values in panel b, it is possible to calculate how incorrect the modeled spin-down is compared with the true physical value (assume that the timing model uses a spin period of  $P_{\text{spin}} = 650$  ms and a period derivative of  $\dot{P}_{\text{spin}} = 4 \times 10^{-14}$  s/s). What is the true spin-down rate? By what percent was the modeled  $\dot{P}_{\text{spin}}$  off from the true value? (this is giving you an idea for how precisely  $\dot{P}_{\text{spin}}$  can be measured) You will need the equation:

$$\phi(t) = \frac{1}{2}\dot{\nu}_{\text{spin}}t^2, \quad (3)$$

where  $\phi$  is the pulsar spin phase as a function of time.

Also note that  $\dot{P}_{\text{spin}} = -\dot{\nu}_{\text{spin}}/\nu_{\text{spin}}^2$ .

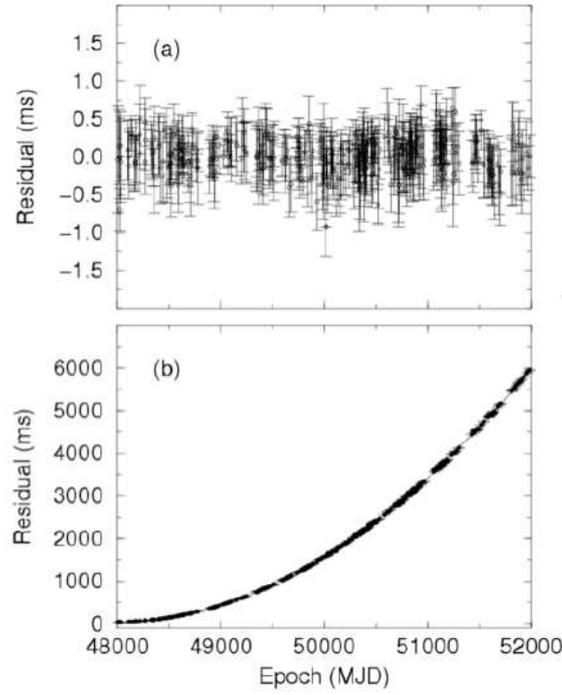


Figure 4: Panel a): Residuals for a pulsar in which all relevant physical processes that affect the times of arrival (TOAs) have been modeled. The individual points are TOAs. Panel b): How the residuals look if the period derivative (equivalently frequency derivative) of the rotation is not well modeled. Note that ‘MJD’ stands for Modified Julian Date, and is just a representation of time, in fractional days, with respect to an arbitrary reference point. From Lorimer & Kramer (2005).

- [C] (4 pts) Consider Figure 5. This is showing the residuals of times of arrival (TOAs) compared to the timing model when the position is not well modeled. A 365-day sinusoid is seen in the pulsar timing residuals. How far off is the position, in degrees? You can assume that the pulsar lies in the Ecliptic plane (orbital plane of the Earth-Sun system).  $c = 300000 \text{ km s}^{-1}$  and the distance to the solar system barycenter (centre of gravity of the solar system) is  $150 \times 10^6 \text{ km}$ .

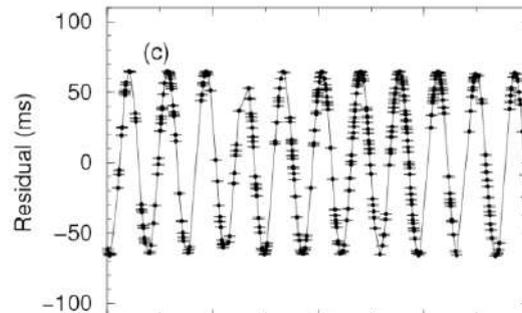


Figure 5: Residuals for a pulsar in which the position is clearly not well modeled. A 365-day sinusoid is seen in the pulsar timing residuals, and this has an amplitude of about 65 ms. From Lorimer & Kramer (2005).

## 6 The Kapitza Pendulum

*Prof. dr. J.S. (Jean-Sébastien) Caux*  
*Institute for Theoretical Physics – University of Amsterdam*  
 10 points

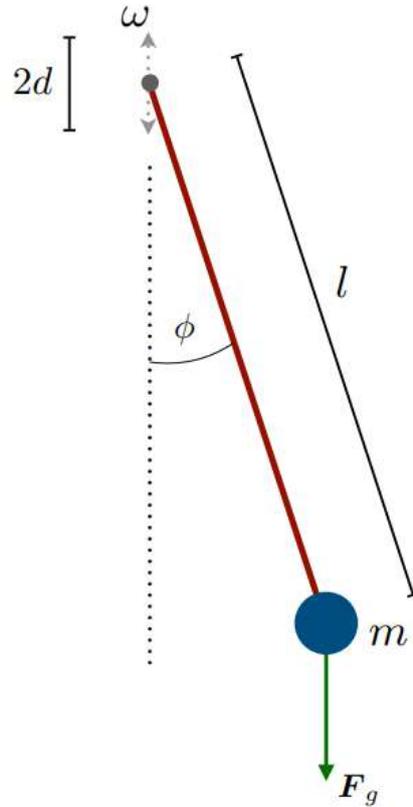


Figure 6: The driven rigid pendulum, known as the Kapitza pendulum.

Consider a pendulum consisting of a rigid arm of length  $l$  (we assume the arm to have negligible mass and to be infinitely rigid), with a mass  $m$  attached at the end (blue dot in the figure).

At a given time  $t$ , the pendulum's angle with respect to the vertical is denoted as  $\phi(t)$  (we assume the pendulum's movement to be restricted to a 2d vertical plane). Gravity with strength  $g$  is acting in the downwards direction.

The suspension point of the pendulum is subjected to periodic sinusoidal driving at frequency  $\omega$  with amplitude  $d \ll l$  in the vertical direction. Explicitly, we force the suspension point  $\mathbf{s}(t)$  of the pendulum to obey

$$\mathbf{s}(t) = (0, d \sin \omega t)$$

where we set the origin of the axes at the  $t = 0$  position of the suspension point.

[A] (2 pts) Write the kinetic energy  $T$  and potential energy  $V$  of the pendulum in terms of the variables  $\phi(t)$  and  $\dot{\phi}(t) \equiv \frac{d\phi(t)}{dt}$ .

[B] (1 pt) The Euler-Lagrange equations of the system are given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

where  $L = T - V$  is the Lagrangian of the pendulum. Show that this results in the following equation of motion:

$$\ddot{\phi} + \frac{\sin \phi}{l} (g - \omega^2 d \sin \omega t) = 0.$$

- [C] (1 pt) We will consider rapid driving of our pendulum, namely we consider the driving frequency  $\omega$  to be much larger than the pendulum's normal (non-driven) oscillation frequency  $\omega_0 \equiv \sqrt{g/l}$ . This leads us to expect that we can separate the movement of the pendulum into fast and slow time dependencies:

$$\phi(t) = \phi_s(t) + \phi_f(t)$$

in which  $\phi_s$  represents slow oscillations, and  $\phi_f$  represents fast ones.

To go further, we need a smart parametrization. Which of the following 3 choices is the right one? Justify your choice.

$$1) \phi_f(t) = \frac{d}{l} \cos \phi_s \sin \omega t, \quad 2) \phi_f(t) = -\frac{d}{l} \sin \phi_s \sin \omega t, \quad 3) \phi_f(t) = \frac{d}{l} \sin \phi_s \sin \omega t.$$

- [D] (2 pts) With the separation into slow and fast modes, the equation of motion can be rewritten (without approximation)

$$\ddot{\phi}_s + \ddot{\phi}_f + \frac{1}{l} (g - \omega^2 d \sin \omega t) \sin(\phi_s + \phi_f) = 0.$$

Exploiting the fact that  $\frac{d}{l} \ll 1$ , and considering low-amplitude, high-frequency scaling (in other words considering  $\omega$  large,  $\frac{d}{l}$  small but  $\omega \frac{d}{l}$  as not necessarily small as compared to  $\omega_0$ ), show that the equation of motion for the slow mode becomes

$$\begin{aligned} \ddot{\phi}_s \left( 1 - \frac{d}{l} \cos \phi_s \sin \omega t \right) + \frac{d}{l} \dot{\phi}_s^2 \sin \phi_s \sin \omega t - 2 \frac{\omega d}{l} \dot{\phi}_s \cos \phi_s \cos \omega t \\ + \frac{g}{l} \sin \phi_s \left( 1 - \frac{d}{l} \cos \phi_s \sin \omega t \right) + \frac{1}{2} \left( \frac{\omega d}{l} \right)^2 \sin 2\phi_s \sin^2 \omega t = 0. \end{aligned}$$

- [E] (2 pts) Show that averaging over the fast oscillations (for example integrating over one “fast” period) leads to the simplified equation

$$\ddot{\phi}_s + \frac{g}{l} \sin \phi_s + \left( \frac{\omega d}{2l} \right)^2 \sin 2\phi_s = 0.$$

- [F] (2 pts) Argue that these equations can be interpreted as coming from a pendulum with effective potential

$$V_{\text{eff}} = -mgl \cos \phi_s - \frac{m}{8} \omega^2 d^2 \cos 2\phi_s.$$

What are the minima of this potential if  $\omega d < \sqrt{2gl}$ ? Are they stable or unstable, namely: is  $\frac{\partial^2 V_{\text{eff}}}{\partial \phi_s^2}$  positive (stable) or negative (unstable)?

What about if  $\omega d > \sqrt{2gl}$ ? Describe what can happen to the pendulum in that case (in other words: list all possible equilibrium configurations and determine whether they are stable or not).

## Epilogue

This wonderful mechanical device (first described in the scientific literature by A. Stephenson in 1908 [1]) is known as the Kapitza pendulum, after Pyotr Kapitza who explained its behaviour in 1951 [2]. If you want to see a working example, you can take a look at this [homemade version](#).

## References

- [1] Stephenson, A.: *XX. On induced stability*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 15:86, 233-236, 1908, doi:[10.1080/14786440809463763](https://doi.org/10.1080/14786440809463763).
- [2] Kapitza, P. L.: *Dynamic stability of a pendulum when its point of suspension vibrates*, Soviet Phys. JETP. 21: 588–597, 1951; Kapitza, P. L.: *Pendulum with a vibrating suspension*, Usp. Fiz. Nauk. 44: 7–15, 1951.

## 7 Topological Mechanical Metamaterials

Dr. C.J.M. (Corentin) Coulais

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10 points

Recent years have witnessed the advent of mechanical analogues of topological insulators in so-called mechanical metamaterials, which are architected materials with superior properties. In this problem, we cover a canonical one-dimensional case, where the physics can be clearly illustrated. Specifically, we will establish the so-called Bulk-Boundary Correspondence (BBC). The BBC is a relation between a property of the periodic bulk material and a property at the edges of the material. The BBC is a very important generic feature of topological materials, which can be found in large swathes of science, from quantum mechanics, to mechanics, acoustics and photonics and with many potential applications, where electronic, mechanical or electromagnetic waves need to be manipulated.

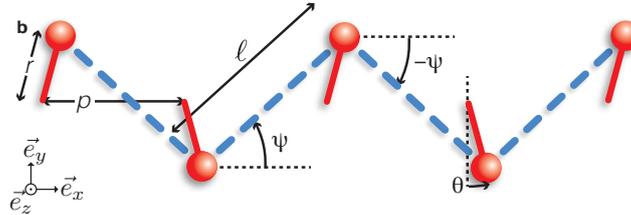


Figure 7: **Canonical model of a topological mechanical metamaterial.** Each rotor  $n$  (red) has a radius  $r$ , tilt angle  $\theta$  with respect to the  $y$ -axis, angular moment of inertia  $J$  and is connected to its neighbors by a spring (blue) of spring constant  $k$ , rest-length  $\ell$  and orientation  $\pm\psi$  with respect to the  $x$ -axis. There are in total  $N$  rotors and  $N - 1$  springs. In this specific sketch  $N = 5$ .

### Equations of motion.

- [A] (2 pts) Demonstrate that the torque on each rotor  $\vec{T}_n$  induced by the elongation its neighbouring springs  $\delta\ell_n$  and  $\delta\ell_{n-1}$  is given by  $\vec{T}_n = k(a\delta\ell_n - b\delta\ell_{n-1})\vec{e}_z$ , where  $a = (p+2r\sin\theta)/\sqrt{p^2 + 4r^2\cos^2\theta}$  and  $b = (p - 2r\sin\theta)/\sqrt{p^2 + 4r^2\cos^2\theta}$  (using Fig. 8a).

Demonstrate that the elongation in each spring  $\delta\ell_n$  is related to  $\delta\theta_n$  and  $\delta\theta_{n+1}$ , the angular displacement of rotors  $n$  and  $n + 1$  with respect to their equilibrium position, as  $\delta\ell_n = -a\delta\theta_n + b\delta\theta_{n+1}$  in the limit where  $\delta\theta_n \ll 1$  and  $\delta\theta_{n+1} \ll 1$  (using Fig. 8b).

Use the previous two results to show that the equations of motion of the structure shown in Fig. 7 are

$$J\frac{d^2\delta\theta_1}{dt^2} = kr^2\cos^2\theta(a^2\delta\theta_1 - ab\delta\theta_2), \quad (4)$$

$$J\frac{d^2\delta\theta_n}{dt^2} = kr^2\cos^2\theta(ab\delta\theta_{n-1} - (a^2 + b^2)\delta\theta_n + ab\delta\theta_{n+1}) \text{ for } n \in [2, N - 1], \quad (5)$$

$$J\frac{d^2\delta\theta_N}{dt^2} = kr^2\cos^2\theta(-ab\delta\theta_{N-1} + b^2\delta\theta_N). \quad (6)$$

Hint 1: use the geometrical constraints for the structure in the undeformed state  $\ell\cos\psi = p$  and  $\ell\sin\psi = 2r\cos\theta$  and their generalisation in the deformed state.

Hint 2: in the limit where  $\delta\alpha \ll 1$ ,  $\cos(\alpha + \delta\alpha) \simeq \cos(\alpha) - \sin(\alpha)\delta\alpha$  and  $\sin(\alpha + \delta\alpha) \simeq \sin(\alpha) + \cos(\alpha)\delta\alpha$ .

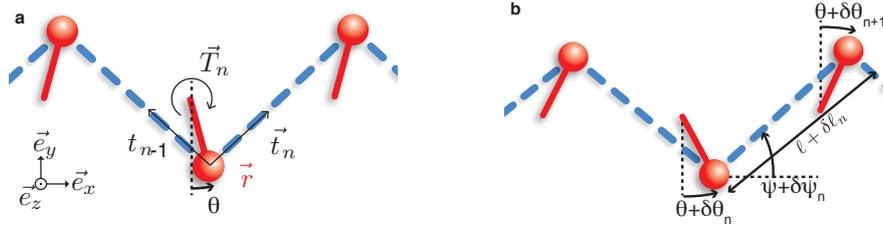


Figure 8: **Mechanical analysis.** (a) Torque balance on unit cell  $n$ . (b) Geometry of the system in the deformed configuration.

### Zero-modes in real space for an open chain.

- [B] (1 pt) What is the solution  $\delta\theta_n$  vs.  $n$  when the elongation of the spring  $\delta\ell_n$  is zero? This solution is called zero-mode.
- [C] (1 pt) Sketch the zero-mode  $\delta\theta_n$  vs.  $n$  for  $a/b < 1$ ,  $a/b = 1$  and  $a/b > 1$ . Discuss the differences between these cases and provide an interpretation of the relation between the geometry of the structure and the shape of the zero-mode.

### Spectrum in Fourier space

- [D] (1 pt) Assuming that plane waves  $\delta\theta_n = \delta\theta_0 \exp i(\omega t - qn)$  are solutions of Eq. (5), show that the frequency can be written in the form of a dispersion relation  $\omega(q) = \pm r \cos \theta \sqrt{k/J} \sqrt{D(q)}$ , where  $D(q) = (a^2 + b^2) - 2ab \cos q$ .
- [E] (1 pt) Sketch the dispersion relation for  $a/b < 1$ ,  $a/b = 1$  and  $a/b > 1$ , respectively. What differs between these three dispersion relations? Provide an interpretation of the relation between the geometry of the structure and the dispersion relation.

### Topological invariant

- [F] (1 pt) Show that the function  $D(q)$  can be factorized as  $D(q) = R(q)Q(q)$  and give the expression of  $R(q)$  and  $Q(q)$ .
- [G] (1 pt) A strict analogy between the present mechanical system and a quantum tight-binding model can be drawn. In this analogy, the Bloch-Hamiltonian can be written as

$$H(q) = \begin{pmatrix} 0 & Q(q) \\ R(q) & 0 \end{pmatrix}. \quad (7)$$

This Hamiltonian describes the dynamics of electrons on the polyacetylene molecule, see Fig. 9. Discuss the relation between the eigenvalues of  $H(q)$  and the frequencies  $\omega(q)$  introduced above.

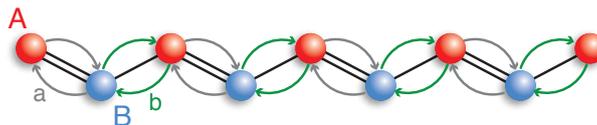


Figure 9: **Quantum analogue of the topological mechanical metamaterial.** Tight-binding model of the polyacetylene molecule, described by the Bloch Hamiltonian in Eq. 7.  $a$  ( $b$ ) is the hopping strength of the electrons of the double (single) bonds.

- [H] (1 pt) Calculate the eigenvectors of  $H(q)$ ,  $|\psi_1(q)\rangle$  and  $|\psi_2(q)\rangle$ .

[I] (1 pt) The Berry connection is defined as  $\mathcal{A}(q) = -i \left\langle \psi(q) \left| \frac{\partial}{\partial q} \psi(q) \right. \right\rangle$  and a topological invariant is defined as  $\nu = (2\pi)^{-1} \int_0^{2\pi} dq \mathcal{A}(q)$ . Show that

$$\mathcal{A}(q) = -\frac{1}{2} \left( \frac{be^{iq}}{a - be^{iq}} + \frac{be^{-iq}}{a - be^{-iq}} \right) \quad (8)$$

and calculate the topological invariant  $\nu$  as a function of  $a/b$ . Discuss the relation between the value of the topological invariant  $\nu$  and the shape of the zero-mode. Hint:  $\int_{\mathcal{C}} dz \frac{1}{z-z_0} = 2i\pi$  if  $|z_0| < 1$  and  $\int_{\mathcal{C}} dz \frac{1}{z-z_0} = 0$  if  $|z_0| > 1$ , where  $\mathcal{C}$  is the unit circle in the complex plane.



## 8 Spin Dynamics Due To Mobile Electron

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 10 points

We consider a system with two spin-1/2 particles on locations  $L$  and  $R$ , and an electron (with spin-1/2) which can hop between both locations, with real hopping amplitude  $h > 0$ . If the electron is located at  $L$ , then there is an interaction energy ( $g > 0$ )

$$H_L = g \vec{S}_L \cdot \vec{s}.$$

If the electron is at location  $R$ , then the energy is

$$H_R = g \vec{S}_R \cdot \vec{s}.$$

Here,  $\vec{S}_L, \vec{S}_R$  are the spin operators of the fixed particles and  $\vec{s}$  is the spin operator for the electron. The system has 16 possible states, which we denote as

$$|m_L, m; m_R\rangle, \quad |m_L; m, m_R\rangle, \quad \text{with } m_L, m, m_R = -1/2, 1/2,$$

where the notation  $m;$  indicates that the electron has location  $L$ , and  $; m$  that its location is  $R$ . A different possible notation is with the help of arrow symbols,

$$|1/2, 1/2; 1/2\rangle \rightarrow |\uparrow, \uparrow; \uparrow\rangle, \quad \text{etc.}$$

The spin operators are of the form  $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ , with  $\vec{\sigma}$  the Pauli matrices

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The  $z$ -components of the spin operators act on the basis states as

$$\begin{aligned} S_L^z |m_L, m; m_R\rangle &= \hbar m_L |m_L, m; m_R\rangle, \\ s^z |m_L, m; m_R\rangle &= \hbar m |m_L, m; m_R\rangle, \\ S_R^z |m_L, m; m_R\rangle &= \hbar m_R |m_L, m; m_R\rangle, \end{aligned}$$

and analogous for the states  $|m_L; m, m_R\rangle$ . From now on, we work with the units for which  $\hbar = 1$ . The kinetic energy of the electron is  $H_K$ , with matrix elements

$$\langle m_L, m; m_R | H_K |m_L; m, m_R\rangle = h, \quad \langle m_L; m, m_R | H_K |m_L, m; m_R\rangle = h.$$

The total Hamiltonian (energy operator) is

$$H = H_L + H_R + H_K.$$

We propose that at  $t = 0$ , the system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow, \downarrow; \uparrow\rangle - |\downarrow, \uparrow; \uparrow\rangle).$$

We now want to calculate the following probabilities: the probability  $P_{\uparrow, \uparrow; \downarrow}(t)$  of finding the state  $|\uparrow, \uparrow; \downarrow\rangle$  and the probability  $P_{\uparrow, \uparrow; \uparrow}(t)$  of finding state  $|\uparrow, \uparrow; \uparrow\rangle$ , at time  $t$ . A straightforward approach involves diagonalising a  $16 \times 16$  matrix. By making use of symmetries it can be done in a smarter way.

[A] (1 pt) For  $t \ll g, t \ll h$ , the leading terms in the probabilities we are looking for are of the form

$$P_{\uparrow, \uparrow; \downarrow}(t) \propto h^a g^b t^m, \quad P_{\uparrow, \uparrow; \uparrow}(t) \propto h^c g^d t^n.$$

Determine, without long calculations, the values  $a, b, m$  and  $c, d$  and  $n$ .

- [B] (3 pts) Show that the total spin  $\vec{S} = \vec{S}_L + \vec{s} + \vec{S}_R$  commutes with  $H$ . Organise the 16 states as eigenstates  $S^2$  and  $S^z$  and give a list of the possible values of  $(S^2, S^z)$ , including their respective multiplicity. Show that  $|\psi(0)\rangle$  belongs to a group of 4 eigenstates with equal values for  $S^2$  en  $S^z$ . What are these values?
- [C] (1 pt) Give a basis  $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$  for the space found at b), with  $e_1 = |\psi(0)\rangle$  and  $e_3$  the similar state for the electron at location  $R$ . Subsequently, write  $H$  as a  $4 \times 4$  matrix with respect to this basis.
- [D] (1 pt) We now also use the symmetry  $P$  (parity), which exchanges  $L$  and  $R$ , and define

$$|e_{1,\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \pm |e_3\rangle), \quad |e_{2,\pm}\rangle = \frac{1}{\sqrt{2}}(|e_2\rangle \pm |e_4\rangle)$$

so that

$$P|e_{i,\pm}\rangle = \pm|e_{i,\pm}\rangle, \quad i = 1, 2.$$

Show that in the basis  $\{|e_{1,+}\rangle, |e_{2,+}\rangle, |e_{1,-}\rangle, |e_{2,-}\rangle\}$ ,  $H$  is a block matrix,

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

with

$$H_{\pm} = \begin{pmatrix} -\frac{3g}{4} \pm h/2 & \pm\sqrt{3}h/2 \\ \pm\sqrt{3}h/2 & \frac{g}{4} \mp h/2 \end{pmatrix}.$$

- [E] (1 pt) Determine the evolution operators  $U_H(t) = \exp(-iHt)$ . Hint: first show that for a Hamiltonian  $M$  of the form

$$M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

the evolution operator  $U_M(t)$  is written as

$$U_M(t) = \exp[-iMt] = \cos(st) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \frac{\sin(st)}{s} \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

with  $s = \sqrt{a^2 + b^2}$ .

- [F] (1 pt) Now give  $|\psi(t)\rangle$  in the basis  $\{|e_{1,+}\rangle, |e_{2,+}\rangle, |e_{1,-}\rangle, |e_{2,-}\rangle\}$ .
- [G] (1 pt) Determine the inner products  $\langle \uparrow, \uparrow; \downarrow | e_{i,\pm} \rangle$ ,  $\langle \uparrow; \uparrow, \downarrow | e_{i,\pm} \rangle$  and calculate  $\langle \uparrow, \uparrow; \downarrow | \psi(t) \rangle$  and  $\langle \uparrow; \uparrow, \downarrow | \psi(t) \rangle$ .
- [H] (1 pt) Now determine the leading terms in  $P_{\uparrow, \uparrow; \downarrow}(t)$  and  $P_{\uparrow; \uparrow, \downarrow}(t)$  for  $t \ll g$ ,  $t \ll h$  and compare the result with your prediction under a).

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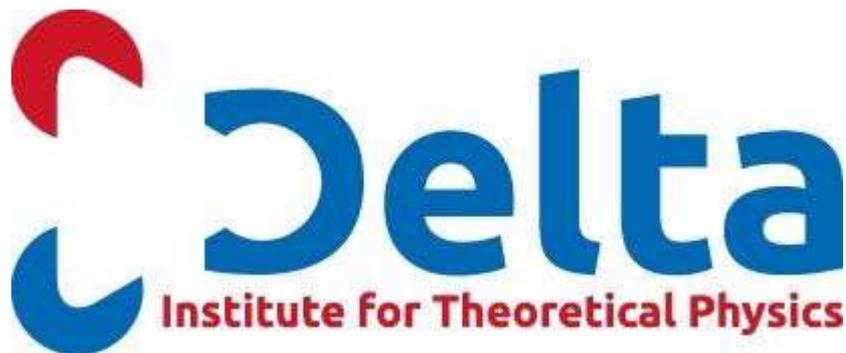
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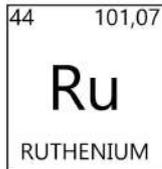
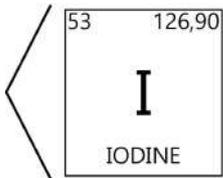
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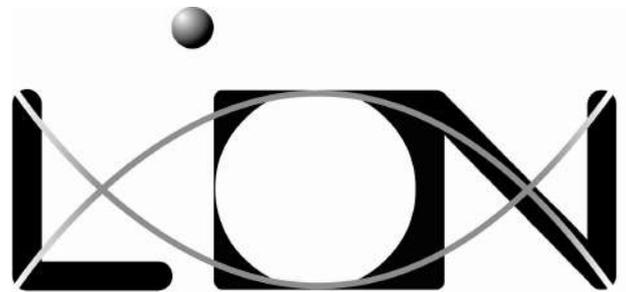


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